OUT(F_n) DOES NOT VIRTUALLY LIFT TO THE HANDLEBODY GROUP

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1. Introduction

Let V_g be a handlebody of genus g, and let $\mathcal{H}_g$ be the *handlebody group*, i.e. the mapping class group of V_g. The action on the fundamental group of V_g defines a natural homomorphism

$$A : \mathcal{H}_g \to \mathrm{Out}(F_g).$$

The purpose of this short note is to prove the following theorem.

Theorem 1.1. *Let $g \geq 4$ be arbitrary. Then A does not split over any finite index subgroup Γ of $\mathrm{Out}(F_g)$. That is, for any finite index subgroup $\Gamma < \mathrm{Out}(F_g)$ there is no homomorphism $s : \Gamma \to \mathcal{H}_g$ so that $A \circ s$ is the identity.*

Note that for $g = 1$ the theorem is obviously false (since there are no outer automorphisms of $\mathbb{Z}$). In genus 2 the theorem is also false: consider a torus T with one boundary component. The trivial interval bundle $T \times [0, 1]$ is a handlebody of genus 2. In this way we get a homomorphism $i : \mathrm{Mcg}(\Sigma_{1,1}) \to \mathcal{H}_2$. But the mapping class group of a once-punctured torus $\Sigma_{1,1}$ is $\mathrm{Out}(F_2)$, and i yields a section to $A : \mathcal{H}_2 \to \mathrm{Out}(F_2)$.

In the case $g = 3$ our methods fail; it is not entirely clear to the author if the result remains true.

2. The proof

We will prove the theorem by contradiction. We begin by picking a free basis $x_1, \ldots, x_g$ of F_g, and identify the subgroup generated by $x_i, i > 1$ once and for all with the free group F_{g-1}. Now, for any $w \in F_{g-1}$ we consider the automorphisms $L_w, R_w : F_g \to F_g$ defined by

$$L_w(x_1) = wx_1, \quad L_w(x_i) = x_i,\ i > 1$$
$$R_w(x_1) = x_1 w, \quad R_w(x_i) = x_i,\ i > 1$$

Note that for any w, w' the automorphisms $L_w, R_{w'}$ commute. Also note that

$$(L_w)^n = L_{w^n}, \quad (R_w)^n = R_{w^n}.$$

To prove Theorem 1.1 we will show that for w, w' chosen suitably, no powers of $L_w, R_{w'}$ admit lifts to $\mathcal{H}_g$ which commute, showing the theorem.

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To choose w, w' we use an fully irreducible non-geometric automorphism $\Theta' : F_{g-1} \to F_{g-1}$ (these exist since $g-1 \geq 3$ by our assumption) and extend it to an automorphism Θ of F_g by setting

$$\Theta(x_1) = x_1, \quad \Theta(x_i) = \Theta'(x_i),\ i > 1.$$

We then have the following facts

(1) If $x \in F_g$ is an element which is fixed up to conjugacy by some power of Θ, then x is conjugate to x_1^n for some $n \in \mathbb{Z}$.
(2) If $F_g = A * B$ or $F_g = A*$ is a one-edge splitting of the free group which is preserved by Θ up to conjugacy, then A (or B in the first case) is conjugate to F_{g-1}.

Recall that a *meridian* is an essential simple closed curve on ∂V_g which bounds a disk in V_g. Alternatively, consider the map

$$p : \pi_1(\partial V_g) \to \pi_1(V_g) = F_g$$

induced by the inclusion of the boundary into the handlebody. A meridian is a simple closed curve in the kernel of p. If α is a meridian, and D is a disk bounded by α, then by the van Kampen theorem D defines a one-egde splitting of F_g. Up to conjugacy, this splitting depends only on the free isotopy class of α, and we call it the splitting defined by α. The following lemma is now immediate from the two facts about Θ above.

Lemma 2.1. *Let $\Psi \in A^{-1}(\Theta)$ be any handlebody group element representing Θ, and suppose $\Psi(\alpha) = \alpha$ for a simple closed curve. Then either $p(\alpha)$ is conjugate to x_1^n for some n, or α is a meridian which defines a free splitting compatible with $\langle x_1 \rangle * F_{g-1}$.*

By definition

$$\Theta \circ L_w \circ \Theta^{-1} = L_{\Theta(w)}$$
$$\Theta \circ R_{w'} \circ \Theta^{-1} = R_{\Theta(w')}$$

Suppose that A admitted a section s over some finite index subgroup $\Gamma < \mathrm{Out}(F_n)$. Up to replacing Θ and w, w' by large enough powers we may assume $\Theta, L_w, R_{w'} \in \Gamma$. Let $\psi = s(\Theta), \lambda = s(L_w), \rho = s(L_{w'})$ be the supposed lifts into $\mathcal{H}_g$. We then have

Lemma 2.2. *If α is a simple closed curve which is fixed both by ψ and λ (or ρ) then α is a meridian defining the co-rank 1 free factor $F_{g-1} < F_g$.*

Proof. If $\psi(\alpha) = \alpha$, then by the previous lemma we have to exclude two possibilities.

- α **maps to x_1^n in $\pi_1(V_g)$ for some $n \neq 0$:** In this case, L_w (or $R_{w'}$) would also fix the conjugacy class of x_1^n, which they clearly do not.
- α **is a meridian and defines the splitting $\langle x_1 \rangle * F_{g-1}$:** In this case, L_w (or $R_{w'}$) would also fix this splitting. However, they map x_1 to an element not conjugate to x_1, therefore this is impossible.

□

Since $L_w, L_{w'}$ commute, the same is true for λ and ρ. In fact, we have

$$\psi^n \lambda \psi^{-n} = s(\Theta)^n s(L_w) s^(\Theta)^{-n} = s(\Theta^n L_w \Theta^n) = s(L_{\Theta^n(w)})$$

and therefore conclude that $\psi^n \lambda \psi^{-n}$ and ρ commute for all n.

L_w and $R_{w'}$ generate a free Abelian subgroup of rank 2 in $\mathrm{Out}(F_g)$, and therefore λ, ρ also generate a free Abelian subgroup of rank 2 in $\mathcal{H}_g$. To exploit this information, we now consider $\mathcal{H}_g$ as a subgroup of the mapping class group $\mathrm{Mcg}(\Sigma_g)$ of a genus g surface (via the restriction homomorphism $\mathcal{H}_g \to \mathrm{Mcg}(\partial V_g)$ which is injective).

As mapping classes of Σ_g ρ, λ cannot be pseudo-Anosov, since pseudo-Anosov mapping classes can never generate a free Abelian group of rank 2. In fact, there will be a multicurve C so that $\rho(C) = C = \lambda(C)$ (the canonical reduction system of the subgroup generated by ρ, λ). Up to passing to powers, we may assume that both λ and ρ preserve each curve in C, and each complementary component of C. Additionally, we may assume that in each non-annular complementary component of C, the restrictions of λ and ρ are either the identity, or pseudo-Anosov maps. Call a region *active* for λ or ρ if it restricts to a pseudo-Anosov there.

Note that there has to be at least one active region, as otherwise λ and ρ would have a power which is equal, which is clearly false for $L_w, R_{w'}$.

Denote the active regions of λ by $Y_1, \ldots, Y_r$, and consider $\psi^k Y_j$. These are the active regions for $\lambda_k = \psi^k \lambda \psi^{-k}$. As the mapping class λ_k commutes with ρ we have that for each j either

- **A:** $\psi^k Y_j$ is contained in some inactive region of ρ for infinitely many k, or
- **B:** $\psi^k Y_j$ is equal to some active region Z of ρ for infinitely many k, and a suitable power $\lambda_k^r|_Z$ commutes with (the pseudo-Anosov map) $\rho|_Z$ as mapping classes of Z.

In both cases, we first observe that some power of ψ fixes a region which is also fixed by ρ – and therefore, by Lemma 2.2, this region is the complement of a single meridian α, which defines the splitting $\langle x_1 \rangle * F_{g-1}$. Thus, we have either

- **A:** ρ is a Dehn twist about α. Or,
- **B:** ρ restricts to a pseudo-Anosov on $\Sigma - \alpha$, and (up to passing to a power) ψ and λ preserve $\Sigma - \alpha$.

The first case is impossible, since $A(\rho)$ is nontrivial, and Dehn twist about meridians act trivially on $\pi_1(V_g)$.

In case B), since λ_k and ρ commute, the pseudo-Anosov ψ on $\Sigma - \alpha$ itself would have to commute with ρ (after possibly passing to powers). Namely, if λ_k is an infinite order element commuting with ρ, it has a power which is a pseudo-Anosov with the same stable and unstable foliations as ρ. But in this case, the stable and unstable foliations of λ_k are the images of those of λ under ψ^k. Hence, ψ^k preserves these foliations and has therefore a power which is equal to a power of λ_k.

However, case B) also cannot occur for *all* choices of w' – since in that case all $R_{w'}$ would have powers which agree, which is clearly false. Therefore both possibilities lead to a contradiction, proving Theorem 1.1.