BIG MAPPING CLASS GROUPS ACTING ON HOMOLOGY

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Abstract. We study the action of (big) mapping class groups on the first homology of the corresponding surface. We give a precise characterization of the image of the induced homology representation.

1. Introduction

Surfaces are among the most basic and most fundamental objects in geometry and topology. Although, as spaces, they may seem simple to understand, their symmetries – mapping classes – certainly are not.

Given a surface S, a first approach to understand its mapping class group $\mathrm{MCG}(S)$ is to consider the natural action on the first homology $\mathrm{H}_1(S;\mathbb{Z})$. This leads to the *homology representation*

$$\rho_S\colon\ \mathrm{MCG}(S) \to \mathrm{Aut}(\mathrm{H}_1(S;\mathbb{Z})).$$

For a surface S of finite genus $g \geq 1$ with at most one puncture, it is well known that the elements in the image of ρ_S are precisely those which preserve the algebraic intersection form $\hat{\imath}$ (first shown by [Bur89], see [FM12, Chapter 6] for a discussion of the result). Usually, this is phrased as saying that $\rho_S\colon\ \mathrm{MCG}(S) \to \mathrm{Sp}(2g;\mathbb{Z})$ is surjective (as $\hat{\imath}$ is a symplectic pairing for such a surface).

In this article, we determine the image of ρ_S for any surface and, in particular, those of infinite type. The first case is that of the Loch Ness monster surface (i.e. the surface of infinite genus and one end). Here, the result is very similar to the closed case, however the standard proofs in the closed case do not directly carry over.

Theorem 1. *Let S be the Loch Ness monster surface. The image of ρ_S is the group of automorphisms of $\mathrm{H}_1(S;\mathbb{Z})$ that preserve the algebraic intersection form.*

As the Loch Ness monster surface is one-ended, $\hat{\imath}$ is symplectic and Theorem 1 is equivalent to saying that the natural homomorphism $\mathrm{MCG}(S) \to \mathrm{Sp}(\mathbb{N};\mathbb{Z})$ is surjective (see Section 3 for more details).

For more general surfaces, the situation is more complicated. For finite-type surfaces, the mapping class group permutes the punctures (and therefore the homology classes they define). For an infinite-type surface, one similarly has to encode the structure of the ends of S in homology to capture the action of the mapping class group on ends. We do this by defining the *homology end filtration* $\mathcal{F}$ of $\mathrm{H}_1(S;\mathbb{Z})$. It consists of the collection of the homologies of unbounded subsurfaces with a single boundary component[1]. Further, for a homology class $[\delta]$ defined by a separating, oriented, simple, closed curve, we denote by $\mathcal{L}([\delta])$ the set of ends of S to the left of δ (this is well-defined by Lemma 2.3).

With this terminology, we can state our main result as follows.

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[1]with an extra technical condition – see Definitions 2.1 and 4.7

Theorem 2. *Let S be an infinite-type surface, different from the Loch Ness monster and the once-punctured Loch Ness monster. If ϕ is an automorphism of $\mathrm{H}_1(S;\mathbb{Z})$ preserving both $\hat{\iota}$ and $\mathcal{F}$, then the following hold:*

i) Exactly one of ϕ and $-\phi$ lies in the image of ρ_S.
ii) ϕ preserves homology classes defined by separating simple closed curves.
iii) ϕ determines a homeomorphism f_ϕ of the space of ends of S, and ϕ lies in the image of ρ_S exactly if

$$f_\phi(\mathcal{L}([\delta])) = \mathcal{L}(\phi([\delta]))$$

for some (hence any) simple separating closed curve δ which is non-trivial in $H_1(S;\mathbb{Z})$.

Actually, we can show that the theorem holds for finite-type surfaces with at least four punctures. Furthermore, we can also characterize the image of ρ_S in the case of the once-punctured Loch Ness monster (see Section 3).

We emphasize that even the proof of Theorem 1 already requires ideas not necessary in the finite-type case. Namely, in the classical case one starts with a collection of simple closed curves α_i, β_i intersecting in a standard pattern and realizes the classes $\phi([\alpha_i]), \phi([\beta_i])$ with the correct intersection pattern; it is then easy to construct a mapping class with the correct action. In the Loch Ness monster case, to follow this approach one also needs to realize the classes $\phi([\alpha_i]), \phi([\beta_i])$ by curves not accumulating in any compact subset of S. To take care of this, we adapt an argument of Richards [Ric63]; the details are discussed in Section 3.

To prove Theorem 2, the first step is to show that (under the given assumption on the surface) ultrafilters of $\mathcal{F}$ are in correspondence with the ends of the surface (Lemma 4.8). It follows that an automorphism ϕ preserving $\mathcal{F}$ induces a permutation of its ultrafilters and hence a map f_ϕ of $\mathrm{Ends}(S)$ (Proposition 4.10).

The second step is to deal with homology classes of separating simple closed curves. We note that two such curves induce the same class in homology if and only if the set of ends to the left of one is the same as the set of ends to the left of the other (Lemma 2.3). Furthermore, we can show that these classes can be detected using $\mathcal{F}$ (Proposition 4.11), which implies that they are permuted by any ϕ satisfying the hypotheses of the theorem.

To finish the proof, we use again a variation of the same argument of Richards that we employ for the Loch Ness monster case. While the structure of this step is the same in both cases, having to deal with more ends renders the proof less transparent.

A natural complement of our study is the investigation of the kernel of ρ_S, called the *Torelli group* of S. For finite-type surface this has been the subject of a sizeable amount of research (the survey [Joh83] gives an excellent overview over the by-now classical theory). In recent years, more progress has been made, and the Torelli group is by now fairly well understood.

Recently, the Torelli group has been investigated for infinite-type surfaces as well by Aramayona, Ghaswala, Kent, McLeay, Tao and Winarski [AGK+19]. Among the results they obtain, they characterize which elements belong to this subgroup by showing that the Torelli group of an infinite-type surface is topologically generated by its compactly-supported elements and hence by separating Dehn twists and bounding pair maps.

1.1. **Necessity of the conditions.**

In this section we will discuss how all the conditions in Theorem 2 are necessary, by providing examples of automorphisms not induced by mapping classes where one of the hypotheses is not satisfied.

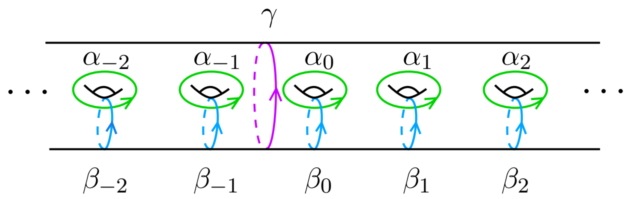

FIGURE 1. Curves for a homology basis of Jacob's ladder

Already finite-type surfaces with punctures show that preserving the algebraic intersection pairing is not sufficient to guarantee realizability. Indeed, mapping classes of the closed genus-g surface with n punctures permute the punctures, and therefore the mapping class group acts on the isotropic subspace as a permutation representation and this fact is not seen by $\hat{\imath}$.

More interesting examples can be constructed on Jacob's ladder surface, i.e. the two-ended infinite-genus surface with no planar ends. We consider the homology basis given by the curves depicted in Figure 1.

Consider the automorphism ϕ_1 fixing $[\gamma]$ and $[\alpha_i], [\beta_i]$ for i even and sending $[\alpha_i], [\beta_i]$ to $[\alpha_{-i}], [\beta_{-i}]$, respectively, for i odd.

The automorphism ϕ_1 cannot be realized by a mapping class because the sequence of curves $\{\alpha_n\}_{n\in\mathbb{N}}$ exit one end, but the representatives of the images under ϕ_1 accumulate to both ends. One can check that that ϕ_1 does not preserve the homology end filtration: more precisely, it can be proved that if we denote by X the subsurface to the left of γ, then $\phi_1(\mathrm{H}_1(X;\mathbb{Z}))$ is not in $\mathcal{F}$. This example shows how the homology end filtration is important to control which ends are accumulated by non-isotropic vectors.

Next, consider the automorphism ϕ_2 which fixes all basis elements except for $[\gamma]$, which is mapped to $-[\gamma]$. Again, ϕ_2 cannot be induced by a mapping class. This time it is because the sequence of curves $\{\alpha_n\}_{n\in\mathbb{N}}$ exit the end to the right of γ, but to the left of any representative of $-[\gamma]$ (e.g. γ with the opposite orientation). In terms of the condition in Lemma 4.14,

$$f_{\phi_2}(\mathcal{L}([\gamma])) \neq \mathcal{L}(\phi_2([\gamma])).$$

On the other hand, $-\phi_2$ is induced by a mapping class (the involution which can be informally described as the rotation of angle π around an axis joining the two ends of S, see Figure 2).

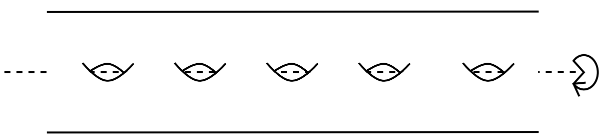

FIGURE 2. A mapping class inducing $-\phi_2$

Note that it is also not enough to require that algebraic intersection and topological type of curves be preserved (where by this we mean that the image of the class of a simple closed curve is the class of a simple closed curve in the same mapping class group orbit), as shown by ϕ_1 and ϕ_2.

Finally, one could wonder if there is a characterization of the image of ρ_S in terms of the set of simple isotropic classes instead of the homology end filtration; we comment on this in Section 4.5.

1.2. **Structure of the paper.** After some preliminaries about surfaces and their homology (Section 2), we deal with the case of the Loch Ness monster with at most one puncture in Section 3. The proof of Theorem 1 contains many of the ideas that are necessary for the general case, but it is simpler since there is only one end.

In section 4 we introduce the main new tool, the homology end filtration and we prove the main result (Theorem 2) in Section 5.

We end the paper with an appendix collecting some realization results for homology and cohomology classes: characterizations of homology classes represented by simple closed curves that the authors could not find in the literature (which may be of independent interest) and a description of which cohomology classes are given by intersection with proper arcs joining two ends.

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2. Preliminaries

Throughout, a *surface* will refer to an oriented, connected, second countable, Hausdorff two-dimensional manifold. Unless stated otherwise, a surface does not have boundary – the one notable exception being subsurfaces of other surfaces. A surface is *of finite type* if its fundamental group is finitely generated and *of infinite type* otherwise.

The *mapping class group* of the surface S is the group of orientation-preserving homeomorphisms of S up to isotopy:

$$\mathrm{MCG}(S) = \mathrm{Homeo}^+(S)/\text{isotopy}.$$

Throughout the article, a *curve* will refer to a simple, closed, oriented curve; in addition, we will routinely conflate the isotopy class of a curve with a representative.

A curve is *essential* if it bounds neither a disk nor a once-punctured disk; it is *separating* if its complement is disconnected and *non-separating* otherwise.

When discussing subsurfaces, we assume that every boundary component is an essential curve with the induced orientation, i.e. such that the subsurface is to the left of the curve.

We will denote by i the *geometric intersection number* of two curves (note: geometric intersection does not take into account the orientation of the curves). A collection of curves $\{\alpha_i, \beta_i\}_{i\in I}$ has the *standard (symplectic) intersection pattern* if $i(\alpha_i, \alpha_j) = 0, i(\beta_i, \beta_j) = 0$, and $i(\alpha_i, \beta_j) = \delta_{ij}$ for all $i, j \in I$.

An *arc* in a surface is the image of a proper embedding of either $(0, 1)$, $[0, 1)$ or $[0, 1]$ into the surface. When a boundary point of the interval is included, the corresponding point on the surface must belong to a boundary component. As with curves, we do not distinguish between an arc and its isotopy class (isotopies of arcs are taken relative to the boundary where appropriate).

2.1. **Ends of a surface.** An *end* of a surface is an equivalence class of a descending chain $U_1 \supset U_2 \supset \dots$ of open connected subsurfaces with compact boundary and such that for any compact K there is an index n_K such that for all $n \geq n_K$, $K \cap U_n = \emptyset$. Two such chains $U_1 \supset U_2 \supset \dots$ and $V_1 \supset V_2 \supset \dots$ are equivalent if for every n there is an N such that $U_N \subset V_n$ and $V_N \subset U_n$.

The *space of ends* $\mathrm{Ends}(S)$ is the set of ends endowed with the topology generated by sets of the form U^*, where U is an open subset with compact boundary, and

$$U^* = \{[U_1 \supset U_2 \supset \dots] \mid \exists n : U_n \subset U\}.$$

An end $[U_1 \supset U_2 \supset \dots]$ is *planar* if there exists an integer n such that U_n is homeomorphic to a subset of the plane (or, equivalently, has genus 0). Otherwise, the end is *non-planar*, and every U_n has infinite genus[2]. An end is *isolated* if it is an isolated point of the space of ends. We will routinely refer to an isolated planar end as a *puncture*.

It is easy to check that $\mathrm{Ends}_g(S)$, the subset of non-planar ends, is a closed subset of $\mathrm{Ends}(S)$.

Kerékjártó and Richards [Ric63] showed that surfaces are topologically classified by the triple $(g, (\mathrm{Ends}(S), \mathrm{Ends}_g(S)))$, where $g \in \mathbb{N} \cup \{0, \infty\}$ is the genus and $(\mathrm{Ends}(S), \mathrm{Ends}_g(S))$ is considered as a pair of topological spaces, up to homeomorphism.

2.2. **Homology of surfaces.** The main focus of this article is the first homology of a surface considered with integral coefficients; accordingly, when referring to the homology of a surface S, we are referring to $\mathrm{H}_1(S; \mathbb{Z})$.

Every homology class in $\mathrm{H}_1(S; \mathbb{Z})$ can be represented by a – possibly non-simple – loop in S. Given a homology class $x \in \mathrm{H}_1(S; \mathbb{Z})$, we say that x is *simple* if there is a simple closed curve α such $[\alpha] = x$. In this case, we say that x is *represented by* α.

The *algebraic intersection number*, denoted $\hat{\imath}$, defines a bilinear, antisymmetric form on $\mathrm{H}_1(S; \mathbb{Z})$. An element x of $\mathrm{H}_1(S; \mathbb{Z})$ is *isotropic* if $\hat{\imath}(x, y) = 0$ for every $y \in \mathrm{H}_1(S; \mathbb{Z})$. If neither complementary component of a separating curve on a non-compact surface has compact closure, then the curve is non-trivial in homology; hence, we see that the form $\hat{\imath}$ is symplectic if and only if $|\mathrm{Ends}(S)| \leq 1$. Note that if x is a simple (non-)isotropic homology class and α is a curve representing x, then α is (non-)separating.

Also note that if a is an arc, algebraic intersection of homology classes with a is a well defined linear functional $\hat{\imath}(a, \cdot) : \mathrm{H}_1(S; \mathbb{Z}) \to \mathbb{Z}$ and hence gives a cohomology class in $H^1(S; \mathbb{Z})$.

Throughout the paper we will be interested into two special types of subsurfaces, *star* and *flare* surfaces.

Definition 2.1. A *star surface* is a connected finite-type subsurface so that all boundary components are separating curves in S and all complementary components are unbounded. A *flare surface* is an unbounded subsurface X with a single boundary component, which is separating, and such that the closure of $S \smallsetminus X$ has at least two ends or infinite genus. The embedding of a star or flare surface X into a surface S induces a monomorphism from the homology $\mathrm{H}_1(X; \mathbb{Z})$ to $\mathrm{H}_1(S; \mathbb{Z})$, and so we will abuse notation and identify $\mathrm{H}_1(X; \mathbb{Z})$ with its image under the above monomorphism.

[2]Sometimes in the literature a non-planar end is also referred to as an end *accumulated by genus*, as every neighborhood has infinite genus.

2.3. **Homology classes of simple closed curves.** In our study, we require some results on the interaction of simple closed curves with homology classes. The first lemma is a criterion to detect simple non-isotropics. The proof is standard (also in the infinite-type setting) and is delegated to Appendix A. As mentioned in the introduction, in the appendix we also collect a number of further results on the interplay between homology and simple representability that are not required for the main argument, but may be of independent interest.

Lemma 2.2. *Let S be any surface and $x \in \mathrm{H}_1(S;\mathbb{Z})$. Then x is a simple non-isotropic if and only if there exists $y \in \mathrm{H}_1(S;\mathbb{Z})$ such that $\hat{\imath}(x,y) = 1$.* □

The complementary components of a separating curve γ in a surface S determine two disjoint clopen sets $\mathcal{L}(\gamma)$ and $\mathcal{R}(\gamma)$ that partition $\mathrm{Ends}(S)$. The sets are labelled so that, when considering the orientation of γ, the set $\mathcal{L}(\gamma)$ consists of the ends to the left of γ and $\mathcal{R}(\gamma)$ those to the right.

We now give a lemma determining when simple separating curves define the same homology class.

Lemma 2.3. *Let S be any surface and let α, β be two separating simple closed curves. Then $[\alpha] = [\beta]$ if and only if $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$.*

Proof. First observe that two ends are on different sides of α exactly if there is an arc connecting these ends so that $\hat{\imath}([\alpha], a) \neq 0$. This shows that homologous curves induce the same decomposition of ends.

Now, suppose the set of ends to the left of α is the same as the set of ends to the left of β. Let Σ be a compact subsurface containing $\alpha \cup \beta$ (where we allow Σ to have boundary components homotopic to punctures), such that all connected components $\{X_i \mid i \in I\}$ of $S \smallsetminus \Sigma$ are unbounded. Since α and β induce the same partition of ends, a surface X_i is to the right of α if and only if it is to the right of β. But then if I_r is the set of indices i such that X_i is to the right of α, we have

$$[\alpha] = \sum_{i \in I_r} \sum_{\gamma \subset \partial X_i} [\gamma] = [\beta].$$

□

As mentioned in the introduction, the previous lemma allows us to define $\mathcal{L}(c)$ for any simple isotropic c: we set $\mathcal{L}(c) = \mathcal{L}(\alpha)$, where α is any separating simple closed curve representing c.

Lemma 2.4. *Let S be any surface and let $X \subset S$ be a subsurface with separating boundary components. Suppose that α is a simple closed curve which is disjoint from X. If $[\alpha] = [\beta]$ for some loop $\beta \subset X$, then $[\alpha] = \pm[\partial_j X]$, where $\partial_j X$ is one of the boundary curves of X.*

Proof. Let $i\colon X \hookrightarrow S$ be the inclusion and i_* the map induced on homology. As α is disjoint from X, we have that $\hat{\imath}([\alpha], v) = 0$ for all $v \in i_* \mathrm{H}_1(X,\mathbb{Z})$ and, as $[\alpha] \in i_* \mathrm{H}_1(X,\mathbb{Z})$, we can conclude that $\hat{\imath}([\alpha], v) = 0$ for all $v \in \mathrm{H}_1(S,\mathbb{Z})$. So, $[\alpha]$ is isotropic and hence α is separating.

Without loss of generality, suppose that X is to the left of α. Now, there exists a unique j such that α is to the right of $\partial_j X$, which implies $\mathcal{L}(\partial_j X) \subseteq \mathcal{L}(\alpha)$. If equality holds, then $[\alpha] = [\partial_j X]$. If equality fails, then there is an arc a in $S \smallsetminus X$ connecting an end in $\mathcal{L}(\alpha) \smallsetminus \mathcal{L}(\partial X)$ to an end in $\mathcal{R}(\alpha)$. It follows that $|\,\hat{\imath}([\alpha], a)| = 1$ and that $\hat{\imath}(x, a) = 0$ for every $x \in i_* \mathrm{H}_1(X;\mathbb{Z})$, which is a contradiction. □

3. The Loch Ness Monster surface

In this section we discuss the case of the Loch Ness monster surface – the infinite-genus surface with a unique end – and of the once-punctured Loch Ness Monster. For these surfaces, the complication of preserving the structure of ends is not necessary, and so the result takes a form very reminiscent of the closed case.

Theorem 3.1. *If S is the Loch Ness monster surface with at most one puncture, then the map $\rho_S : \mathrm{MCG}(S) \to \mathrm{Aut}(\mathrm{H}_1(S;\mathbb{Z}))$ is a surjection onto the group of automorphisms of homology preserving the algebraic intersection form and acting as the identity on the isotropic subspace.*

Note that for the Loch Ness monster the isotropic subspace is trivial, so we recover Theorem 1.

Proof. We first prove the result for the Loch Ness monster L, where the condition on the action on the isotropic subspace is void.

Clearly, a mapping class preserves algebraic intersection, so we just need to prove that if ϕ is an automorphism preserving $\hat{\imath}$, then it is induced by a mapping class.

Fix a compact exhaustion $\{\Sigma_n\}_{n\in\mathbb{N}}$ of L, where Σ_n has genus n and connected boundary. We want to construct two sequences of subsurfaces $\{A_n\}$ and $\{B_n\}$, each with connected boundary, and homeomorphisms $f_n\colon A_n \to B_n$ such that:

(1) $\Sigma_n \subset A_n$ for every odd n and $\Sigma_n \subset B_n$ for every even n,
(2) $f_n|_{A_{n-1}} = f_{n-1}$, and
(3) the induced homomorphism $(f_n)_*\colon H_1(A_n;\mathbb{Z}) \to H_1(B_n;\mathbb{Z})$ agrees with $\phi\big|_{\mathrm{H}_1(A_n;\mathbb{Z})}$.

Note that condition (1) implies that both sequences $\{A_n\}$ and $\{B_n\}$ are exhaustions. Therefore, using condition (2) implies that we can take the direct limit[3] of the f_n, and the resulting map f is a homeomorphism of L. Condition (3) then implies that f acts as ϕ on homology.

We construct the desired sequence of subsurfaces via induction.

Base case: Set $g_1 = 1$ and $A_1 = \Sigma_1$. Choose a geometric homology basis α_1, β_1 of $\mathrm{H}_1(\Sigma_1;\mathbb{Z})$ and realize the image classes $\phi([\alpha_1]), \phi([\beta_1])$ by non-separating curves α_1', β_1' intersecting once (as in [FM12, Theorem 6.4]). Let B_1 be the one-holed torus obtained by taking a regular neighborhood of $\alpha_1' \cup \beta_1'$ and let f_1 be a homeomorphism between A_1 and B_1 sending α_1 to α_1' and β_1 to β_1'.

Induction step: Suppose that we are given A_n, B_n and f_n satisfying conditions (1)-(3) above.

If n is even, set $A_{n+1} = \Sigma_m$, where $m \geq n+1$ is such that $A_n \subset \Sigma_m$ and A_n is not homotopic to Σ_m. Set g_{n+1} to be the genus of A_{n+1}. Choose curves $\alpha_{g_n+1}, \ldots, \beta_{g_{n+1}}$ in $A_{n+1} \smallsetminus A_n$ with the standard intersection pattern. Note that the images $\phi([\alpha_i])$, $\phi([\beta_i])$, for $i > g_n$, belong to $\mathrm{H}_1(S \smallsetminus B_n;\mathbb{Z})$, as they have algebraic intersection zero with all vectors in a basis for $\mathrm{H}_1(B_n;\mathbb{Z})$. Hence, as the B_n have a single boundary component, the classes can be realized by curves α_i', β_i' outside B_n and with the standard intersection pattern. Let B_{n+1} be a genus g_{n+1} surface with one boundary component containing B_n and all the curves constructed. $B_{n+1} \smallsetminus B_n$ and $A_{n+1} \smallsetminus A_n$ are both surfaces with genus

[3] Formally, we view $\{A_n\}$ and $\{B_n\}$ as directed systems with respect to inclusion and, as both sequences are exhaustions of S, both of their direct limits are exactly S. It is in this setting that we use the universal property of direct limits to obtain the map f.

$g_{n+1} - g_n$ with two boundary components, so we can extend f_n to a homeomorphism f_{n+1} sending the α_i, β_i, for $g_n + 1 \leq i \leq g_{n+1}$ to the corresponding α'_i, β'_i.

If n is odd, the argument is similar: set $B_{n+1} = \Sigma_m$, where $m \geq n+1$ is such that $B_n \subset \Sigma_m$ is not homotopic to Σ_m. Proceed identically to the above case switching the roles of A_n, α_i, and β_i with those of B_n, α'_i, and β'_i, respectively, in every instance. Now after constructing A_{n+1}, we extend $f_n^{-1}\colon B_n \to A_n$ to a homeomorphism $h\colon B_{n+1} \to A_{n+1}$ mapping α'_i, β'_i, for $g_n + 1 \leq i \leq g_{n+1}$, to the corresponding α_i, β_i. We finish by setting $f_{n+1} = h^{-1}$.

In the case of the once-punctured Loch Ness monster surface L', we take an exhaustion of finite-type surface $\{\Sigma_n\}_{n\in\mathbb{N}}$ so that Σ_n has genus n, connected boundary, and contains the unique puncture of L'. The same proof then yields the result. □

Remark 3.2. The role of alternating between constructing A_n and B_n is a bit subtle: the main purpose is that doing so allows us to simultaneously build both f and f^{-1}. If we only constructed the A_n, we would not be able to guarantee that the resulting map f is a homeomorphism: the issue is that there are non-surjective embeddings of infinite-type surfaces into themselves and such maps can arise as direct limits. In this case, the boundary curves of the images of the A_n would have to accumulate in S. Therefore, one should view this alternating technique as a means to avoid this accumulation issue. Note that this technique appears in Richards's paper on the classification of surfaces [Ric63].

3.1. **The infinite-degree integral symplectic group.** Consider an infinite-rank $\mathbb{Z}$-module V with a countable basis $\{a_i, b_i \mid i \in \mathbb{N}\}$ and a symplectic form ω such that for every $i, j \in \mathbb{N}$

$$\omega(a_i, b_j) = \delta_{i,j}$$
$$\omega(a_i, a_j) = \omega(b_i, b_j) = 0.$$

The *infinite-degree integral symplectic group* $\mathrm{Sp}(\mathbb{N};\mathbb{Z})$ is the group of linear automorphisms of V preserving ω. It is clear that the group of automorphisms of $\mathrm{H}_1(L;\mathbb{Z})$ preserving $\hat{\imath}$ is isomorphic to $\mathrm{Sp}(\mathbb{N};\mathbb{Z})$. Under this isomorphism, we have the immediate corollary of Theorem 1:

Corollary 3.3. *If L is the Loch Ness monster surface, then the action of* $\mathrm{MCG}(L)$ *on* $H_1(L;\mathbb{Z})$ *induces an epimorphism* $\mathrm{MCG}(L) \to \mathrm{Sp}(\mathbb{N};\mathbb{Z})$. □

We endow $\mathrm{Sp}(\mathbb{N};\mathbb{Z})$ with the topology whose subbasis is given by sets of the form

$$U_v = \{A \in \mathrm{Sp}(\mathbb{N},\mathbb{R}) \mid Av = v\}$$

and their left translates. This topology, often referred to as the *permutation topology*, turns $\mathrm{Sp}(\mathbb{N};\mathbb{Z})$ into a topological group. We also consider $\mathrm{MCG}(L)$ as topological group by endowing it with the quotient topology coming from $\mathrm{Homeo}^+(L)$ equipped with the compact-open topology. Using the curve graph, this topology on $\mathrm{MCG}(L)$ can also be described as a permutation topology (see [APVar, Section 2.4] for details). In particular, one can readily show that the homomorphism $\mathrm{MCG}(L) \to \mathrm{Sp}(\mathbb{N};\mathbb{Z})$ is continuous.

For any g, we can naturally embed $\mathrm{Sp}(2g;\mathbb{Z})$ in $\mathrm{Sp}(\mathbb{N};\mathbb{Z})$; this is accomplished by making any element of $\mathrm{Sp}(2g;\mathbb{Z})$ act on the first $2g$ basis vectors and extending it to the identity on the other basis vectors. Similarly, we have natural inclusions of $\mathrm{Sp}(2g;\mathbb{Z})$ in $\mathrm{Sp}(2g';\mathbb{Z})$ for every $g \leq g'$. This gives us a directed system and we can consider the direct limit $\mathrm{Sp}(2\infty;\mathbb{Z}) = \varinjlim \mathrm{Sp}(2g;\mathbb{Z})$, which is a proper subgroup of $\mathrm{Sp}(\mathbb{N};\mathbb{Z})$.

The obvious analogy is to consider the directed system of mapping class groups of surfaces $S_{g,1}$ of genus g with one boundary component. A mapping class is *compactly supported* if it can be represented by a homeomorphism which is the identity outside of a compact set.

The direct limit $\varinjlim \mathrm{MCG}(S_{g,1})$ is the subgroup of compactly supported mapping classes $\mathrm{MCG}_c(L)$ of the Loch Ness monster.

The above discussion yields the following commutative diagram of topological groups:

$$\begin{tikzcd}
\mathrm{MCG}(S_{g,1}) \arrow[r, hook] \arrow[d] & \mathrm{MCG}_c(L) \arrow[r, hook] \arrow[d] & \mathrm{MCG}(L) \arrow[d] \\
\mathrm{Sp}(2g;\mathbb{Z}) \arrow[r, hook] & \mathrm{Sp}(2\infty;\mathbb{Z}) \arrow[r, hook] & \mathrm{Sp}(\mathbb{N};\mathbb{Z})
\end{tikzcd}$$

where all maps are continuous. For the Loch Ness monster, $\mathrm{MCG}_c(L)$ is dense in $\mathrm{MCG}(L)$ [PV18, Theorem 4]. Therefore, the surjectivity of $\mathrm{MCG}(L) \to \mathrm{Sp}(\mathbb{N};\mathbb{Z})$ tells us that $\mathrm{Sp}(2\infty,\mathbb{Z})$ is dense in $\mathrm{Sp}(\mathbb{N};\mathbb{Z})$. (Following the the proof of [PV18, Theorem 4], one can prove this directly as well.)

Corollary 3.4. $\mathrm{Sp}(2\infty,\mathbb{Z})$ *is dense in* $\mathrm{Sp}(\mathbb{N};\mathbb{Z})$. □

3.2. **The action on the quotient of homology by its isotropic subspace.** For any surface S of positive genus, $\hat{\imath}$ induces a symplectic form on the quotient[4] of $\mathrm{H}_1(S;\mathbb{R})$ by the isotropic subspace ${\mathrm{H}_1}^{\mathrm{is}}(S;\mathbb{R})$. As any mapping class preserves $\hat{\imath}$, it preserves the isotropic subspace and so we get an induced representation

$$\bar{\rho}_S : \mathrm{MCG}(S) \to \mathrm{Aut}(\mathrm{H}_1(S;\mathbb{Z})/\,{\mathrm{H}_1}^{\mathrm{is}}(S;\mathbb{Z}),\omega),$$

where the target group is the group of $\mathbb{Z}$-module endomorphisms of the quotient preserving ω, i.e.

- $\mathrm{Sp}(2g;\mathbb{Z})$, if S has positive, finite genus, or
- $\mathrm{Sp}(\mathbb{N};\mathbb{Z})$, if S has infinite genus.

As discussed above for the case of the Loch Ness monster, $\mathrm{MCG}(S)$ has a natural topology, and in this topology both the homomorphisms ρ_S and $\bar{\rho}_S$ are continuous. As a corollary of Theorem 1, we get:

Corollary 3.5. *For any surface S, the homomorphism $\bar{\rho}_S$ is surjective if and only if S has at most one non-planar end.*

Proof. Let $\hat{S}$ be the surface of the same genus as S with no planar ends and such that $\mathrm{Ends}_g(\hat{S})$ is homeomorphic to $\mathrm{Ends}_g(S)$, i.e. $\hat{S}$ is obtained from S by filling in the planar ends of S. It is then possible to embed S into $\hat{S}$, which yields a natural forgetful surjective map $F : \mathrm{MCG}(S) \to \mathrm{MCG}(\hat{S})$ and a natural isomorphism of symplectic spaces

$$\eta : (\mathrm{H}_1(S;\mathbb{Z})/\,{\mathrm{H}_1}^{\mathrm{is}}(S;\mathbb{Z}),\omega) \to (\mathrm{H}_1(\hat{S};\mathbb{Z}),\hat{\imath}).$$

The map $\bar{\rho}_S$ is the composition $\eta^{-1} \circ \rho_{\hat{S}} \circ F$. If S has at most one non-planar end, $\hat{S}$ is either closed or the Loch Ness monster, so $\rho_{\hat{S}}$ is surjective and hence $\bar{\rho}_S$ is as well.

Conversely, if S has at least two non-planar ends, we can generalize the construction of the map ϕ_1 in Section 1.1 to get an explicit example of automorphism which is not in the image of $\bar{\rho}_S$. □

[4] If the genus of S is zero, the quotient is trivial.

4. The homology end filtration

In this section we introduce an extra structure associated to the homology of a surface, called the homology end filtration. Its main purpose is to capture the necessary information of the space of ends of the surface. This structure is a poset of a class of submodules of $\mathrm{H}_1(S;\mathbb{Z})$ whose space of ultrafilters will correspond to the space of ends of the surface. This will give us a way to associate a self map of $(\mathrm{Ends}(S), \mathrm{Ends}_g(S))$ to an automorphism of homology preserving the homology end filtration.

Throughout this section we routinely require an additional condition on a surface, which we denote $(\star)$ and is defined as follows:

> A surface satisfies $(\star)$ if it is either planar with at least 4 ends; of finite positive genus with at least 3 ends; or infinite-genus and not homeomorphic to either the Loch Ness monster or the once-punctured Loch Ness monster surface.

Equivalently, a surface satisfies $(\star)$ if it is not homeomorphic to a surface in the following countable list (which only has finitely many members in a given genus and only two among infinite-type surfaces): a thrice-punctured sphere, a closed surface, a closed surface with one or two points removed, the Loch Ness monster surface, or the Loch Ness monster surface with a point removed.

4.1. Flare surfaces and their homology.

With our orientation conventions and notation, a *flare surface* can be equivalently defined as an unbounded subsurface X whose boundary is a single separating simple closed curve and such that $\mathcal{R}(\partial X)$ is neither empty nor a single planar end. Note that by definition of flare surface, its boundary is non-trivial in homology. Let $\mathcal{FS}$ be the set of all flare surfaces.

The main reason why we are interested in these subsurfaces is the following consequence of the definition of the space of ends.

Lemma 4.1. *Given a surface S satisfying $(\star)$, the set*

$$\{\mathcal{L}(\partial X) \mid X \in \mathcal{FS}\}$$

is a subbasis for $\mathrm{Ends}(S)$ *consisting of clopen sets.* □

As a flare surface X in S is a closed subset of S, the inclusion $X \hookrightarrow S$ is a proper map and hence induces a map $\mathrm{Ends}(X) \to \mathrm{Ends}(S)$; moreover, the fact that the boundary of X is connected guarantees that this map is injective. In particular, it is a homeomorphism onto its image, which allows us to naturally identify $\mathrm{Ends}(X)$ with $\mathcal{L}(\partial X)$.

A consequence of Lemma 4.1 and of the fact that the ends space is Hausdorff is the following:

Lemma 4.2. *Suppose that Y is a flare surface with at least three ends and $e, e' \in \mathcal{L}(\partial Y)$ are distinct elements. Then there is a flare surface $X \subset Y$ so that $e \in \mathcal{L}(\partial X)$ and $e' \notin \mathcal{L}(\partial X)$.* □

We now show that inclusion of homologies of flare surfaces gives inclusion of the corresponding spaces of ends.

Lemma 4.3. *If X and Y are two flare surfaces such that* $\mathrm{H}_1(X;\mathbb{Z}) \leq \mathrm{H}_1(Y;\mathbb{Z})$*, then $\mathcal{L}(\partial X) \subseteq \mathcal{L}(\partial Y)$. Moreover, if* $\mathrm{H}_1(X;\mathbb{Z}) = \mathrm{H}_1(Y;\mathbb{Z})$*, then $\mathcal{L}(\partial X) = \mathcal{L}(\partial Y)$.*

Proof. For a contradiction, suppose that $\mathcal{L}(\partial X) \not\subset \mathcal{L}(\partial Y)$.

First suppose that $\mathcal{L}(\partial X) \smallsetminus \mathcal{L}(\partial Y)$ has at least two elements and let e_1 and e_2 be two such ends. Using that $\partial X \cup \partial Y$ is compact, $\mathcal{L}(\partial X) \smallsetminus \mathcal{L}(\partial Y)$ is clopen, and $\mathrm{Ends}(S)$ is Hausdorff, there exist simple separating closed curves γ_1 and γ_2 in $X \smallsetminus Y$ such that $e_i \in \mathcal{R}(\gamma_i)$, $\gamma_1 \cap \gamma_2 = \emptyset$ and $e_i \notin \mathcal{R}(\gamma_j)$ if $i \neq j$. As $[\gamma_i] \in \mathrm{H}_1(X, \mathbb{Z}) < \mathrm{H}_1(Y, \mathbb{Z})$, by Lemma 2.4 we have that $[\gamma_i] = \pm[\partial Y]$ for $i \in \{1, 2\}$. But as $e_i \in \mathcal{R}(\partial Y)$, we get $[\gamma_1] = [\partial Y] = [\gamma_2]$, which is impossible since $\mathcal{L}(\gamma_1) \neq \mathcal{L}(\gamma_2)$.

So, we may assume that $\mathcal{L}(\partial X) \smallsetminus \mathcal{L}(\partial Y)$ contains a single end, call it e. Repeating the same argument, we can find a simple separating closed curve γ contained in $X \smallsetminus Y$ such that $e \in \mathcal{R}(\gamma)$. Since $|\mathcal{L}(\partial X) \smallsetminus \mathcal{L}(\partial Y)| = 1$, we have that $\mathcal{R}(\gamma) = \{e\}$. Again, we find $[\gamma] = [\partial Y]$ implying $\mathrm{Ends}(S) \smallsetminus \mathcal{L}(\partial Y) = \{e\}$. By the definition of flare surface, e must be non-planar; hence, $X \smallsetminus Y$ has infinite genus and $\mathrm{H}_1(X, \mathbb{Z})$ cannot be a subspace of $\mathrm{H}_1(Y, \mathbb{Z})$, a contradiction. $\square$

This lemma is the motivation for requiring that $\mathcal{R}(\partial X)$ not be a single puncture for a flare surface X. Indeed, if we allowed this, we could, for instance, construct flare surfaces with the same homology but different spaces of ends, as Figure 3 shows.

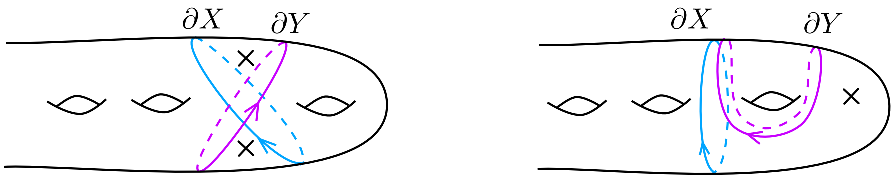

FIGURE 3. Two pairs of (non-flare) surfaces with a single boundary component and the same homology, but different spaces of ends.

We also note that two disjoint flare surfaces that do not cover the entire space of ends have homologies that intersect trivially:

Lemma 4.4. *If X and Y are disjoint flare surfaces such that $\mathcal{L}(\partial X) \cup \mathcal{L}(\partial Y) \neq \mathrm{Ends}(S)$, then $\mathrm{H}_1(X; \mathbb{Z}) \cap \mathrm{H}_1(Y; \mathbb{Z}) = \{0\}$. Moreover, $\mathcal{L}(\partial X) \cup \mathcal{L}(\partial Y) = \mathrm{Ends}(S)$ if and only if $[\partial X] = -[\partial Y]$*

Proof. Suppose $x \in \mathrm{H}_1(X; \mathbb{Z}) \cap \mathrm{H}_1(Y; \mathbb{Z})$. Then it must be an isotropic vector: since it can be realized in X, it must pair to zero with all vectors that can be realized outside of X, but at the same time it can be realized in Y and hence it must pair to zero with all verctors of $\mathrm{H}_1(X; \mathbb{Z})$. Now we can choose a compact subsurface K with (possibly peripheral) separating boundary components containing ∂X and ∂Y in its interior, and such that $x \in \mathrm{H}_1(K; \mathbb{Z})$. Additionally, we choose K so that there is a single component of ∂K contained in $K \smallsetminus (X \cup Y)$. Let

$$\partial K = \{\gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_m, \eta\}$$

where $\gamma_1, \ldots, \gamma_n$ are curves in X, $\delta_1, \ldots, \delta_m$ are curves in Y and η is the curve outside $X \cup Y$. Note that η is homologous to $[\partial X] + [\partial Y]$.

The classes $[\gamma_1], \ldots [\gamma_n]$ form a basis for the isotropic subspace of $\mathrm{H}_1(K \cap X; \mathbb{Z})$, the classes $[\delta_1], \ldots, [\delta_m]$ a basis for the isotropic subspace of $\mathrm{H}_1(K \cap Y; \mathbb{Z})$, and all together they form a basis for the isotropic subspace of $\mathrm{H}_1(K; \mathbb{Z})$. As x must be written at the same time as a linear combination of the $[\gamma_i]$ and as a linear combination of the $[\delta_j]$, it must be the zero vector.

The second part of lemma is now a direct consequence of Lemma 2.3. $\square$

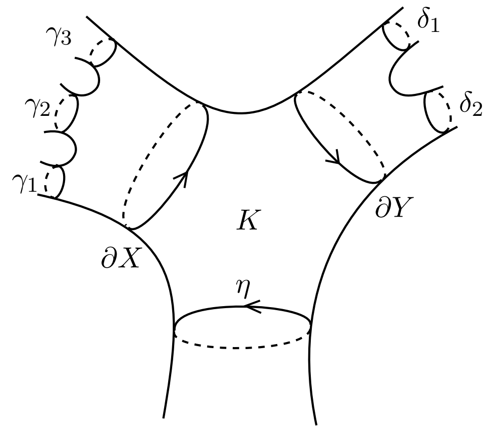

FIGURE 4. Disjoint flare surfaces whose homologies have trivial intersection and the subsurface K in the proof of Lemma 4.4.

We end this section by showing how nesting at the homology level can be translated into geometric nesting in the case of flare surfaces. Since complicated mapping classes can act trivially on the homology of the surface, geometrically intersecting surfaces can have nested homologies. However, we will show that we can find a nested flare surface with the correct homology.

We prove first this type of result in the finite-type case.

Lemma 4.5. *Let K be a finite-type surface and $X', Y \subset K$ two subsurfaces each cut off by a single separating curve (not homotopic to boundary components), so that $\partial K \cap X' \subset Y$, each puncture of X' is a puncture of Y, and $\mathrm{H}_1(X';\mathbb{Z}) < \mathrm{H}_1(Y;\mathbb{Z})$.*
Then there is a subsurface $X \subset K$ bounded by a single curve such that $X \subset Y$, $\partial K \cap X = \partial K \cap X'$ and $\mathrm{H}_1(X;\mathbb{Z}) = \mathrm{H}_1(X';\mathbb{Z})$.

Proof. To simplify the notation, replace all punctures by boundary components.

Let $\gamma_1, \dots, \gamma_r$ be the boundary components of K contained in X'. Let g denote the genus of X'. For $i \leq g$, choose $a_i, b_i \in \mathrm{H}_1(X';\mathbb{Z})$ so that $\hat{\imath}(a_i, b_j) = \delta_{ij}$ and $\hat{\imath}(a_i, a_j) = \hat{\imath}(b_i, b_j) = 0$ for all $i, j \in \{1, \dots, g\}$. Observe that

$$\mathrm{H}_1(X';\mathbb{Z}) = \mathrm{Span}\{a_i, b_i, \gamma_j : i \leq g, j \leq r\}.$$

Let α_i and β_i be simple closed curves in Y homologous to a_i and b_i, respectively, and whose geometric intersection is the same as the algebraic intersection of the corresponding classes. Fix a surface $Z \subset Y$ with a single boundary component containing the α_i and β_i and such that

$$\mathrm{H}_1(Z;\mathbb{Z}) = \mathrm{Span}\{a_i, b_i : i \leq g\}.$$

Choose pairwise disjoint simple arcs $\delta_1, \dots, \delta_r$ contained in $Y \setminus Z$ such that δ_i connects ∂Z and γ_i. Define X to be a regular neighborhood of $Z \cup \bigcup_{i=1}^{r} (\gamma_i \cup \delta_i)$. By construction, X satisfies the requirements. □

Proposition 4.6. *If X', Y are a flare surfaces such that $\mathrm{H}_1(X';\mathbb{Z}) < \mathrm{H}_1(Y;\mathbb{Z})$, then there is a flare surface X with $\mathrm{H}_1(X;\mathbb{Z}) = \mathrm{H}_1(X';\mathbb{Z})$ and $X \subset Y$.*

Proof. Let $K \subset S$ be a star surface which contains $\partial X' \cup \partial Y$. Denote by $U_1, \dots, U_k$ the complementary components of K. Observe that each U_i is either contained in, or disjoint from, X' as they are disjoint from ∂X (and analogously for Y).

Observe that if $U_i \subset X'$, then $\mathrm{H}_1(U_i;\mathbb{Z}) \leq \mathrm{H}_1(X';\mathbb{Z}) \leq \mathrm{H}_1(Y;\mathbb{Z})$; hence, by Lemma 4.3, $\mathcal{L}(\partial U_i) \subset \mathcal{L}(\partial Y)$. It follows by the choice of K that $U_i \subset Y$.

So, up to reordering, we have:

$$X' = U_1 \cup \cdots U_r \cup K_{X'}$$

$$Y = U_1 \cup \ldots U_{r+s} \cup K_Y$$

where $K_{X'} = K \cap X'$ and $K_Y = K \cap Y$. Note also that all punctures of $K_{X'}$ are punctures of K_Y as well since $\mathcal{L}(\partial X') \subset \mathcal{L}(\partial Y)$.

We now want to show that $\mathrm{H}_1(K_{X'};\mathbb{Z}) < \mathrm{H}_1(K_Y;\mathbb{Z})$.

Since we have seen that $\partial U_j \subset K_Y$ for all $j \leq r$ and that all punctures of $K_{X'}$ are also punctures of K_Y, we know that every isotropic vector in $\mathrm{H}_1(K_{X'};\mathbb{Z})$ is also in $\mathrm{H}_1(K_Y;\mathbb{Z})$. Look now at any non-isotropic vector $v \in \mathrm{H}_1(K_{X'},\mathbb{Z}) < \mathrm{H}_1(K;\mathbb{Z})$. Choose a standard basis for homology of K such that the non-separating curves are either completely contained in K_Y or in $K \setminus K_Y$ and all boundaries of K_Y are part of the basis. If we decompose v with respect to this basis, we get

$$v = x + y$$

where $x \in \mathrm{H}_1(K_Y;\mathbb{Z})$ and y is a linear combination of classes of curves in $K \setminus K_Y$. If y were not isotropic, it would have non-zero intersection with some curve in $K \setminus K_Y \subset Y$ and hence so would v, a contradiction since $v \in \mathrm{H}_1(Y;\mathbb{Z})$. So

$$y = \sum_{i=r+s+1}^{k} c_i[\partial U_i] + \sum_{i=k+1}^{k+p} c_i[\gamma_i],$$

where p is the number of punctures in $K \setminus K_Y$ and each γ_i is a curve surrounding one puncture of $K \setminus K_Y$ (and leaving it to the right).

If all c_i are the the same, then y is a multiple of ∂Y and hence belongs to $\mathrm{H}_1(K_Y;\mathbb{Z})$ and so does v. Otherwise there is an arc $\alpha \in K \setminus K_Y$ that intersects y non-trivially and hence it intersects v non-trivially, a contradiction.

So also all non-isotropic vectors of $\mathrm{H}_1(K_{X'};\mathbb{Z})$ belong to $\mathrm{H}_1(K_Y;\mathbb{Z})$, which shows that $\mathrm{H}_1(K_{X'};\mathbb{Z}) < \mathrm{H}_1(K_Y;\mathbb{Z})$.

Note that all boundary components of K that are in $K_{X'}$ are in K_Y as well. This implies that we can apply Lemma 4.5 to find a subsurface $K_X \subset K$ cut off by a single curve, contained in K_Y, with $\partial K \cap K_X = \partial U_1 \cup \cdots \cup \partial U_r$ and

$$\mathrm{H}_1(K_X;\mathbb{Z}) = \mathrm{H}_1(K_{X'};\mathbb{Z}).$$

Hence

$$X = U_1 \cup \cdots U_r \cup K_X$$

is the desired subsurface. □

4.2. **The homology end filtration and its ultrafilters.** The following is the central object of this section.

Definition 4.7. We define

$$\mathcal{F} = \{V < \mathrm{H}_1(S;\mathbb{Z}) \,|\, V = \mathrm{H}_1(X,\mathbb{Z}) \text{ for some } X \in \mathcal{FS}\}$$

and for every $e \in \mathrm{Ends}(S)$ we define $\mathcal{F}_e \subset \mathcal{F}$ to be

$$\mathcal{F}_e = \{V < \mathrm{H}_1(S;\mathbb{Z}) \,|\, V = \mathrm{H}_1(X,\mathbb{Z}) \text{ for some } X \in \mathcal{FS} \text{ with } e \in \mathcal{L}(\partial X)\}.$$

We call $\mathcal{F}$ the *homology end filtration* and we say that an automorphism of $\mathrm{H}_1(S;\mathbb{Z})$ *preserves* $\mathcal{F}$ if it induces a permutation of $\mathcal{F}$.

We emphasize that $\mathcal{F}$ contains only the homology group, without the data of which flare surface yielded the group. Note that $\mathcal{F}$ is endowed with a natural partial order given by inclusion and if ϕ is an automorphism of $\mathrm{H}_1(S;\mathbb{Z})$ preserving the homology end filtration, then it induces an automorphism of $\mathcal{F}$ as a poset.

We first want to show that if an automorphism of $\mathrm{H}_1(S;\mathbb{Z})$ preserves the homology end filtration, then it induces a permutation of the set $\{\mathcal{F}_e \mid e \in \mathrm{Ends}(S)\}$. This will allow us to define an associated map of the space of ends.

To get the result, we will show how these subsets of $\mathcal{F}$ correspond to ultrafilters in $\mathcal{F}$. Recall that, if $(P, \leq)$ is a poset, a *filter* is a non-empty subset F of P such that:

(1) for all $x, y \in F$ there exists $z \in F$ with $z \leq x, z \leq y$;
(2) if $x \in F$ and $x \leq y$, then $y \in F$.

A filter U is called an *ultrafilter* if it is a maximal proper filter of P, that is, $U \neq P$ and if F is a proper filter such that $U \subseteq F$, then $F = U$.

First, we discuss the homology end filtration and its ultrafilters.

Lemma 4.8. *Given a surface S satisfying $(\star)$, U is an ultrafilter if and only if $U = \mathcal{F}_e$ for some $e \in \mathrm{Ends}(S)$.*

Proof. We show first that for every e, $\mathcal{F}_e$ is an ultrafilter.

Let $V, W \in \mathcal{F}_e$ and let X and Y be flare surfaces such that $V = \mathrm{H}_1(X;\mathbb{Z})$ and $W = \mathrm{H}_1(Y;\mathbb{Z})$. The intersection $X \cap Y$ contains a flare surface – say T – with e as an end. Then $\mathrm{H}_1(T;\mathbb{Z}) \in \mathcal{F}_e$ and $\mathrm{H}_1(T;\mathbb{Z}) \leq V$, $\mathrm{H}_1(T;\mathbb{Z}) \leq W$. So property (1) of a filter holds.

Property (2) follows from Lemma 4.3.

Finally, suppose there exists a proper filter U containing $\mathcal{F}_e$. Let $V \in U \smallsetminus \mathcal{F}_e$ and choose a flare-surface X so that $\mathrm{H}_1(X;\mathbb{Z}) = V$ and hence $e \notin \mathcal{L}(\partial X)$. By the assumption on the topology of S, we can find a flare surface Y containing e and disjoint from X. Property (1) of a filter guarantees then the existence of a flare surface Z such that

$$\mathrm{H}_1(Z;\mathbb{Z}) \subset \mathrm{H}_1(X;\mathbb{Z}) \cap \mathrm{H}_1(Y;\mathbb{Z})$$

contradicting Lemma 4.3.

Conversely, let U be an ultrafilter and consider $\mathcal{L}_U = \{\mathcal{L}(\partial X) \mid \mathrm{H}_1(X;\mathbb{Z}) \in U\}$. Property (1) of filters together with Lemma 4.3 implies that the intersection of any finite collection of sets in $\mathcal{L}_U$ is non-empty (i.e. $\mathcal{L}_U$ has the finite intersection property). Hence, as each element of $\mathcal{L}_U$ is closed and $\mathrm{Ends}(S)$ is compact, the intersection $\bigcap_{C \in \mathcal{L}_U} C$ is non-empty. If e is an element in the intersection, then $U \subset \mathcal{F}_e$; hence, by maximality, $U = \mathcal{F}_e$. □

4.3. **A homeomorphism of the space of ends.** Let $\mathcal{U}(\mathcal{F})$ be the set of ultrafilters of $\mathcal{F}$ and, for each $V \in \mathcal{F}$, let

$$N_V = \{U \in \mathcal{U}(\mathcal{F}) \mid V \in U\}.$$

We define a topology on $\mathcal{U}(\mathcal{F})$ by declaring the sets of the form N_V to be a basis. By Lemma 4.8 and since different ends define different ultrafilters, we have a bijective map $\theta\colon \mathrm{Ends}(S) \to \mathcal{U}(\mathcal{F})$ defined by $\theta(e) = \mathcal{F}_e$.

Lemma 4.9. *For a surface S satisfying $(\star)$, the map θ is a homeomorphism.*

Proof. Fix $V \in \mathcal{F}$ and let $N = N_V$. We can then choose $X \in \mathcal{FS}$ such that $V = \mathrm{H}_1(X;\mathbb{Z})$. Tracing definitions, we have that

$$\begin{aligned}\theta^{-1}(N) &= \{e \in \mathrm{Ends}(S) \mid \mathcal{F}_e \in N\}\\ &= \{e \in \mathrm{Ends}(S) \mid V \in \mathcal{F}_e\}\\ &= \{e \in \mathrm{Ends}(S) \mid e \in \mathcal{L}(\partial X)\}\\ &= \mathcal{L}(\partial X),\end{aligned}$$

where the third equality is a consequence of Lemma 4.3. Therefore, θ is a continuous bijective map; moreover, the above chain of equalities (in reverse) shows that θ is an open map and hence a homeomorphism. □

Ultrafilters of $\mathcal{F}$ are preserved under poset automorphisms of $\mathcal{F}$ and, as a consequence, any such automorphism of $\mathcal{F}$ will induce a homeomorphism of $\mathcal{U}(\mathcal{F})$. Consequently, given an automorphism ϕ of $\mathrm{H}_1(S;\mathbb{Z})$ preserving the homology end filtration $\mathcal{F}$, we can define the homeomorphism $f_\phi\colon\ \mathrm{Ends}(S) \to \mathrm{Ends}(S)$ by $f_\phi(e) = \theta^{-1} \circ \hat{\phi} \circ \theta$, where $\hat{\phi}\colon\ \mathcal{U}(\mathcal{F}) \to \mathcal{U}(\mathcal{F})$ is the homeomorphism of $\mathcal{U}(\mathcal{F})$ defined by $\hat{\phi}(N_V) = N_{\phi(V)}$. We record this in the following proposition:

Proposition 4.10. *Let S be a surface satisfying ($\star$). An automorphism ϕ of $\mathrm{H}_1(S;\mathbb{Z})$ preserving the homology end filtration induces a homeomorphism f_ϕ of $\mathrm{Ends}(S)$ defined by the property $\mathcal{F}_{f_\phi(e)} = \hat{\phi}(\mathcal{F}_e)$.* □

4.4. **The homology end filtration and simple isotropics.** As we saw in the last section, an automorphism of the homology end filtration induces a homeomorphism on the space of ends. Given the correspondence between simple isotropics and clopen subsets of the end space, we expect that any automorphism of the homology end filtration must preserve the set of simple isotropics; indeed:

Proposition 4.11. *Let S be a surface satisfying ($\star$). If ϕ is an automorphism of $\mathrm{H}_1(S;\mathbb{Z})$ preserving the homology end filtration, then ϕ preserves the set of simple isotropic elements of $\mathrm{H}_1(S;\mathbb{Z})$.*

To prove this, we need the following lemma:

Lemma 4.12. *Let S be a surface satisfying ($\star$) and let $e \in \mathrm{Ends}(S)$. If e is isolated, then*

$$\bigcap_{V \in \mathcal{F}_e} V = \mathrm{Span}(c)$$

where $c \in \mathrm{H}_1(S;\mathbb{Z})$ is a simple isotropic with $\mathcal{L}(c) = \{e\}$. Moreover, e is isolated if and only if

$$\bigcap_{V \in \mathcal{F}_e} V \neq \{0\}.$$

Proof. Suppose first that e is isolated. Let $V \in \mathcal{F}_e$ and let X be a flare surface such that $V = \mathrm{H}_1(X;\mathbb{Z})$ and $e \in \mathcal{L}(\partial X)$. Then there is a separating simple closed curve $\alpha \subset X$ such that $\mathcal{L}(\alpha) = \{e\}$. Then by Lemma 2.3, $c = [\alpha]$ and thus $\mathrm{Span}(c) \subset V$. Now let us show that the intersection is not bigger than the span of c. There are three cases:

(1) S is planar, in which case by ($\star$) there exists a subsurface Σ with a single puncture corresponding to e, three boundary components, each of which is separating (and not necessarily essential), and unbounded complementary components,

(2) S is non-planar but e is planar, in which case by ($\star$) there exists a genus-1 subsurface Σ with a single puncture corresponding to e, two boundary components, each of which is separating (and not necessarily essential), and unbounded complementary components, or
(3) e is non-planar, in which case by ($\star$) there exists an infinite-genus subsurface Σ with one end corresponding to e, one boundary component, and an unbounded complementary component.

In each case we can construct flare surfaces whose homology groups intersect in the span of c (see Figure 5):

- in case (1), $\mathrm{H}_1(X_1;\mathbb{Z}) \cap \mathrm{H}_1(X_2;\mathbb{Z}) = \mathrm{Span}(c)$;
- in case (2), $\mathrm{Span}(c)$ is the intersection of the homologies of X, of the flare surface with boundary $T_\alpha(\partial X)$, and of the flare surface with boundary $T_\beta(\partial X)$ (where T_γ denotes the Dehn twist about a curve γ);
- in case (3), $\mathrm{Span}(c) = \bigcap_{n\geq 1} \mathrm{H}_1(Y_n;\mathbb{Z})$.

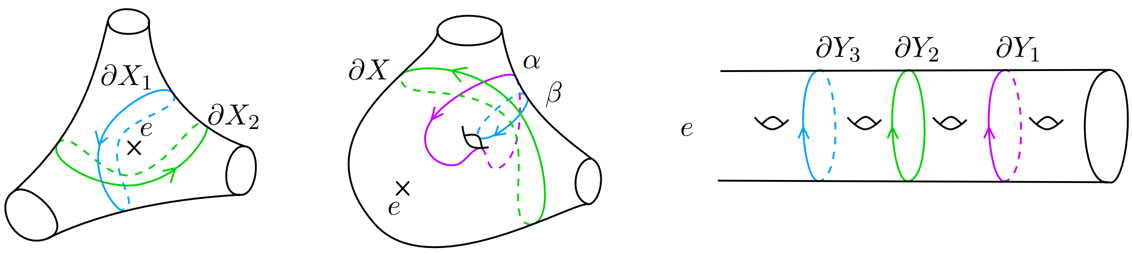

FIGURE 5. The cases when e is isolated and planar (left and center) or non-planar (right)

Suppose now that e is not isolated and let $x \in \bigcap\{V \mid V \in \mathcal{F}_e\}$. Then x can be realized by some loop α in a compact subsurface K with separating boundary curves. Let X be a flare surface containing e with $|\mathrm{Ends}(S) \smallsetminus \mathcal{L}(\partial X)| \geq 2$ and disjoint from K. Furthermore, let $Y \subset X$ be another flare surface with $\mathcal{L}(\partial X) \smallsetminus \mathcal{L}(\partial Y) \neq \emptyset$ and $e \in \mathcal{L}(\partial Y)$. Then $x \in \mathrm{H}_1(S \smallsetminus X;\mathbb{Z})$, because we can realize it in K, and $x \in \mathrm{H}_1(Y;\mathbb{Z})$ since x is in the homology of all flare surfaces containing e. By Lemma 4.4, $\mathrm{H}_1(S \smallsetminus X;\mathbb{Z}) \cap \mathrm{H}_1(Y;\mathbb{Z}) = \{0\}$; hence, $x = 0$. □

Proof of Proposition 4.11. Let $a \in \mathrm{H}_1(S;\mathbb{Z})$ be a simple isotropic. First suppose that S is Jacob's ladder. In this case, $\pm a$ are the unique primitive homology classes contained in the homology of every flare surface; hence, $\phi(a) = \pm a$ and ϕ preserves the unique simple isotropic element.

We can now assume that S is not Jacob's ladder. Now suppose first that neither $\mathcal{L}(a)$ nor $\mathcal{R}(a)$ is a single isolated puncture. Then there is a flare surface X with $[\partial X] = a$. Let Y be the closure of the complement of X in S, so that Y is a flare surface satisfying $[\partial Y] = -a$. The intersection $\mathrm{H}_1(X;\mathbb{Z}) \cap \mathrm{H}_1(Y;\mathbb{Z})$ is generated by a. Note that for every end e of S, either $\mathrm{H}_1(X;\mathbb{Z}) \in \mathcal{F}_e$ and $\mathrm{H}_1(Y;\mathbb{Z}) \notin \mathcal{F}_e$ or vice versa.

As ϕ induces a homeomorphism on the space of ends (Proposition 4.10), it follows that for every end e of S, either $\phi(\mathrm{H}_1(X;\mathbb{Z})) \in \mathcal{F}_e$ and $\phi(\mathrm{H}_1(Y;\mathbb{Z})) \notin \mathcal{F}_e$ or vice versa. Let X' and Y' be such that $\mathrm{H}_1(X';\mathbb{Z}) = \phi(\mathrm{H}_1(X;\mathbb{Z}))$ and $\mathrm{H}_1(Y';\mathbb{Z}) = \phi(\mathrm{H}_1(Y;\mathbb{Z}))$. We know that $\mathrm{H}_1(X';\mathbb{Z}) \cap \mathrm{H}_1(Y';\mathbb{Z})$ is cyclic and generated by $\phi(a)$. Further, $\mathrm{Ends}(S) = \mathcal{L}(\partial X') \sqcup \mathcal{L}(\partial Y')$ implying that $[\partial Y'] = -[\partial X']$. It follows from Lemma 2.4 that in fact

$\mathrm{H}_1(X';\mathbb{Z}) \cap \mathrm{H}_1(Y';\mathbb{Z})$ is generated by $[\partial X']$. Therefore, $\phi(a) = \pm[\partial X']$ and hence is a simple isotropic.

If $\mathcal{L}(a)$ or $\mathcal{R}(a)$ is a single isolated puncture, say p, then

$$\mathrm{Span}(a) = \bigcap_{V \in \mathcal{F}_p} V,$$

hence

$$\mathrm{Span}(\phi(a)) = \bigcap_{V \in \mathcal{F}_{f_\phi(p)}} V.$$

By Lemma 4.12, $\phi(a)$ is a simple isotropic. □

The proof of Proposition 4.11 yields two additional lemmas that we record.

Lemma 4.13. *Let S be a surface satisfying $(\star)$ and let ϕ be an automorphism of* $\mathrm{H}_1(S;\mathbb{Z})$ *preserving the homology end filtration. If X and Y are flare surfaces satisfying*

$$\phi(\mathrm{H}_1(X;\mathbb{Z})) = \mathrm{H}_1(Y;\mathbb{Z}),$$

then $\phi([\partial X]) = \pm[\partial Y]$. □

Lemma 4.14. *Let S be a surface satisfying $(\star)$. If ϕ is an automorphism of* $\mathrm{H}_1(S;\mathbb{Z})$ *preserving the homology end filtration and a is a simple isotropic, then either*

$$\mathcal{L}(\phi(a)) = f_\phi(\mathcal{L}(a))$$

or

$$\mathcal{R}(\phi(a)) = f_\phi(\mathcal{R}(a)).$$

□

In the next lemma, we see that the homeomorphism of the space of ends induced by an automorphism of the homology end filtration either preserves the notion of "to the left" or reverses it coherently across all simple isotropics.

Lemma 4.15. *Let S be a surface satisfying $(\star)$ and let ϕ be an automorphism of* $\mathrm{H}_1(S;\mathbb{Z})$ *preserving the homology end filtration. If there exists a simple isotropic a such that $\mathcal{L}(\phi(a)) = f_\phi(\mathcal{L}(a))$, then $\mathcal{L}(\phi(b)) = f_\phi(\mathcal{L}(b))$ for every simple isotropic b in* $\mathrm{H}_1(S;\mathbb{Z})$.

Proof. We proceed by contradiction: suppose there exists b such that $f_\phi(\mathcal{L}(b)) = \mathcal{R}(\phi(b))$. We have two cases: either $\mathcal{L}(a) \cap \mathcal{L}(b) = \emptyset$ or $\mathcal{L}(a) \cap \mathcal{L}(b) \neq \emptyset$. In the first case, $a+b$ is a simple istropic. It follows that in this case $\mathcal{L}(\phi(a)) \cap \mathcal{L}(\phi(b)) \neq \emptyset$, but then $\phi(a+b) = \phi(a) + \phi(b)$ is not a simple isotropic, which contradicts Proposition 4.11.

In the second case, we choose simple isotropics a' and b' such that $\mathcal{L}(a') = \mathcal{L}(a) \smallsetminus \mathcal{L}(b)$ and $\mathcal{L}(b') = \mathcal{L}(b) \smallsetminus \mathcal{L}(a)$. Observe that $a - a'$ is a simple isotropic. If $f_\phi(\mathcal{L}(a')) = \mathcal{R}(\phi(a'))$, then

$$\mathcal{R}(\phi(a')) = f_\phi(\mathcal{L}(a')) \subset f_\phi(\mathcal{L}(a)) = \mathcal{L}(\phi(a));$$

hence, $\phi(a - a') = \phi(a) - \phi(a')$ is not a simple isotropic, again contradicting Proposition 4.11. Therefore, $f_\phi(\mathcal{L}(a')) = \mathcal{L}(\phi(a'))$ and, by a similar argument, $f_\phi(\mathcal{L}(b')) = \mathcal{L}(\phi(b'))$. This puts us back in the first case and leads us to another contradiction. □

Together, Lemma 4.13 and Lemma 4.15, yield the following:

Lemma 4.16. *Let S be a surface satisfying $(\star)$ and let ϕ be an automorphism of* $\mathrm{H}_1(S;\mathbb{Z})$ *preserving the homology end filtration. If X and Y are flare surfaces satisfying*

$$\phi(\mathrm{H}_1(X;\mathbb{Z})) = \mathrm{H}_1(Y;\mathbb{Z}),$$

then $\phi([\partial X]) = [\partial Y]$. □

We now strengthen Proposition 4.10 by detecting the topological types of ends.

Proposition 4.17. *Let S be a surface satisfying $(\star)$. If ϕ is an automorphism of* $\mathrm{H}_1(S;\mathbb{Z})$ *preserving the homology end filtration, then the homeomorphism f_ϕ of* $\mathrm{Ends}(S)$ *induced by ϕ preserves the set of planar ends.*

Proof. Observe that e is planar if and only if there exists $H \in \mathcal{F}_e$ such that H is an isotropic subspace of $\mathrm{H}_1(S;\mathbb{Z})$. By Proposition 4.11, we have that $\phi(H)$ is isotropic if and only if H is isotropic; hence if e is planar, then so is $f_\phi(e)$. □

4.5. **Characterizing the image of ρ_S using simple isotropic elements.** As we have seen that if a map preserves the filtration, then it preserves the set of simple isotropic vectors, it is natural to ask whether the converse holds. More generally, we can ask if the homology end filtration requirement in Theorem 2 can be replaced by a condition on the action of the automorphism on the simple isotropic classes.

By considering a finite-type surface with genus g and n punctures (for $n \geq 3$), it is easy to see that preserving the algebraic intersection form and the set of simple isotropic classes does not suffice to guarantee realizability: a mapping class cannot send a curve surrounding a single puncture to a curve surrounding two.

Nevertheless, in certain cases one can use the set of simple isotropic classes to describe the image of the homology representation. Namely, we can define a partial order on this set, where we say that $[\gamma] \leq [\delta]$ if $\mathcal{L}(\gamma) \subset \mathcal{L}(\delta)$. Lemma 2.3 shows that this order is well defined. If we ask that an automorphism preserves $\hat{\imath}$ and the poset of simple isotropic classes, the example mentioned above is ruled out. In fact, one could show – with techniques similar to those we use – that this is enough to characterize automorphisms induced by mapping classes in the case of surfaces with at least two ends and at most one non-planar end. On the other hand, as soon as there is more than one non-planar end, this characterization does not hold: for instance, consider Jacob's ladder with the basis depicted in Figure 1. The automorphism fixing all non-isotropic basis vectors and sending $[\gamma]$ to $-[\gamma]$ preserves $\hat{\imath}$ and the poset of simple isotropic classes, but it cannot be given by a mapping class.

5. Proof of the main theorem

The goal of this section is to prove Theorem 2, characterizing the image of the mapping class group in the group of automorphisms of $\mathrm{H}_1(S;\mathbb{Z})$. We will show:

Theorem 5.1. *Let S be either a planar surface with at least four ends, of finite positive genus with at least three ends, or an infinite-genus surface different from the Loch Ness monster and the once-punctured Loch Ness monster. The image of ρ_S consists of the automorphisms $\phi \in \mathrm{Aut}(\mathrm{H}_1(S;\mathbb{Z}))$ preserving both $\hat{\imath}$ and $\mathcal{F}$ and such that there is a simple isotropic class c for which*

$$f_\phi(\mathcal{L}(c)) = \mathcal{L}(\phi(c)).$$

Using this, the main theorem (which we now recall) easily follows.

Theorem 2. *Let S be either a finite-type surface with at least four punctures or an infinite-type surface different from the Loch Ness monster and the once-punctured Loch Ness monster. If ϕ is an automorphism of* $\mathrm{H}_1(S;\mathbb{Z})$ *preserving both $\hat{\imath}$ and $\mathcal{F}$, then the following hold:*

i) Exactly one of ϕ and $-\phi$ lies in the image of ρ_S.
ii) ϕ preserves homology classes defined by separating simple closed curves.

iii) *ϕ determines a homeomorphism f_ϕ of the space of ends of S, and ϕ lies in the image of ρ_S exactly if*

$$f_\phi(\mathcal{L}([\delta])) = \mathcal{L}(\phi([\delta]))$$

for some (hence any) simple separating closed curve δ which is non-trivial in $H_1(S;\mathbb{Z})$.

Proof. Part ii) follows from Proposition 4.11 and Theorem 5.1.

The fact that ϕ induces a homeomorphism of the space of ends is given by Proposition 4.10. This together with Theorem 5.1 yields part iii).

Part i) follows from iii) using Lemma 4.15. □

Let us then prove Theorem 5.1.

Proof of Theorem 5.1. We prove the theorem for infinite-type surfaces. In the case of a finite-type surface, we just need to adapt the base case below.

It is clear that any mapping class induces an automorphism satisfying the conditions in the statement, so we want to show that an automorphim with these properties is induced by a mapping class. Let ϕ be such an automorphism.

Fix an exhaustion Σ_k of S by star surfaces such that no component of $\partial\Sigma_{k+1}$ is homotopic to any component of $\partial\Sigma_k$.

The goal is to construct two nested sequences of star surfaces $\{A_k\}$ and $\{B_k\}$, together with homeomorphisms $f_k\colon A_k \to B_k$ such that:

(1) $\Sigma_k \subset A_k$ for every odd k and $\Sigma_k \subset B_k$ for every even k;
(2) $f_k|_{A_{k-1}} = f_{k-1}$;
(3) f_k induces $\phi\big|_{\mathrm{H}_1(A_k;\mathbb{Z})}$.

As in the proof of Theorem 1, condition (2) implies the direct limit of the f_n exists and condition (1) implies that both sequences are exhaustions and hence $f = \varinjlim f_n$ is a homeomorphism of S. Condition (3) then guarantees that f acts on homology as ϕ.

We will construct surfaces and maps satisfying the additional condition:

(4) For every component X of $S \setminus A_k$ the following holds: if Y is the component of $S \setminus B_k$ bounded by $f_k(\partial X)$, then $\phi(\mathrm{H}_1(X;\mathbb{Z})) = \phi(\mathrm{H}_1(Y;\mathbb{Z}))$.

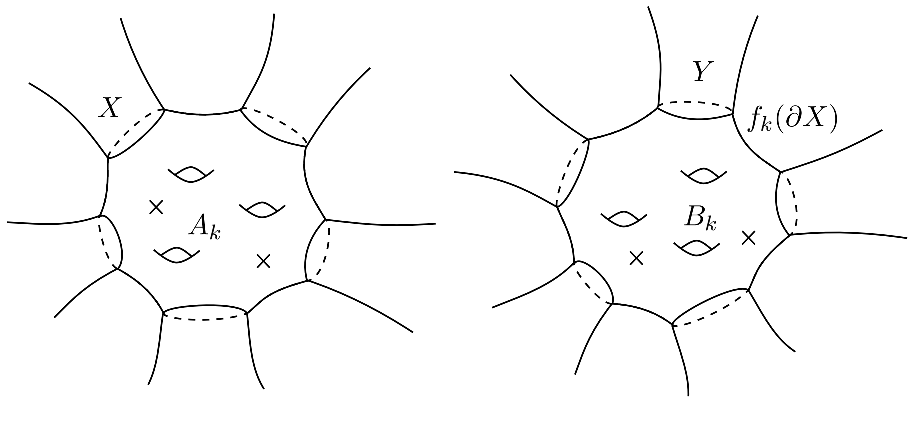

FIGURE 6. The subsurfaces in condition (4)

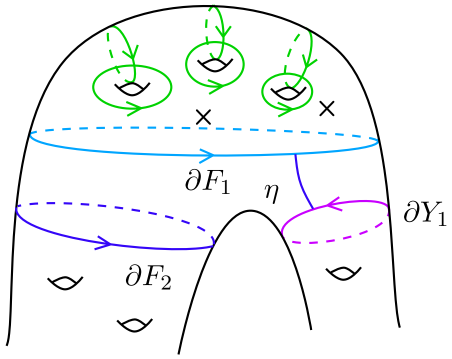

FIGURE 7. Constructing the Y_j

Base case: Let $A_1 = \Sigma_l$ where l is the first index so that Σ_i has either more than one boundary component, or contains more than one puncture. That such an index exists follows from the fact that S is neither the Loch Ness nor the once-punctured Loch Ness surface. Let g_1 be the genus of A_1.

Choose $2g_1$ non-separating curves $\alpha_1, \beta_1, \dots, \alpha_{g_1}, \beta_{g_1}$ with the standard symplectic intersection pattern. Realize the classes $\phi([\alpha_i]), \phi([\beta_i])$ by non-separating curves α_i', β_i' with the standard symplectic intersection pattern (if $g_1 = 0$, we do not do anything in this step).

Let $f_\phi\colon \operatorname{Ends}(S) \to \operatorname{Ends}(S)$ given by Proposition 4.10. Choose a subsurface F_1 of genus g_1 with one boundary component and containing the α_i', β_i' as well as the images via f_ϕ of the punctures of A_1 (if $g_1 = 0$ and A_1 has no punctures, just set $F_1 = \emptyset$).

Denote by $X_1, \dots, X_b$ the complementary components of $A_1 = \Sigma_1$. By construction, for every $j \in \{1, \dots, b\}$, we have $\phi(\mathrm{H}_1(X_j;\mathbb{Z})) \subset \mathrm{H}_1(S \smallsetminus F_1;\mathbb{Z})$. Moreover, if $j \neq j'$, then $\phi(\mathrm{H}_1(X_j;\mathbb{Z})) \cap \phi\,\mathrm{H}_1(X_{j'};\mathbb{Z})$ is trivial, unless $b = 2$, in which case the intersection is generated by $\phi([\partial X_1])$. The goal is to realize these homology groups by disjoint flare surfaces: by Proposition 4.6 there exists a flare surface Y_1 contained in $S \smallsetminus F_1$ such that $\mathrm{H}_1(Y_1;\mathbb{Z}) = \phi(\mathrm{H}_1(X_1;\mathbb{Z}))$. Choose a simple arc η in the closure of $S \smallsetminus F_1$ connecting ∂F_1 and ∂Y_1 and define F_2 to be a regular neighborhood of $F_1 \cup \eta \cup Y_1$ (if $g_1 = 0$ and A_1 has no punctures, just set $F_2 = Y_1$). As $\phi(\mathrm{H}_1(X_2,\mathbb{Z})) \subset \mathrm{H}_1(S \smallsetminus F_2;\mathbb{Z})$, we can find a second flare surface $Y_2 \subset S \smallsetminus F_2$ with $\phi(\mathrm{H}_1(X_2;\mathbb{Z})) = \mathrm{H}_1(Y_2;\mathbb{Z})$. Repeating this process, we obtain $Y_1, \dots, Y_b$ such that $Y_j \cap Y_{j'} = \emptyset$ whenever $j \neq j'$ and such that $\mathrm{H}_1(Y_j;\mathbb{Z}) = \phi(\mathrm{H}_1(X_j;\mathbb{Z}))$ for every $j \in \{1, \dots, b\}$.

Define $B_1 = S \smallsetminus \left(\bigcup_{j=1}^b Y_j\right)$. Then, as $\mathcal{L}(\partial Y_j) = f_\phi(\mathcal{L}(\partial X_j))$ by Lemma 4.16, we have

$$\bigcup \mathcal{L}(\partial Y_i) \cup f_\phi(\{\text{punctures of } A_1\}) = \operatorname{Ends}(S)$$

and hence B_1 is a star surface with b boundary components and as many punctures as A_1. Now, since B_1 and A_1 have the same number of boundary components, the same number of punctures, and isomorphic homology, we can conclude that B_1 is homeomorphic to A_1. Now choose a homeomorphism $f_1\colon A_1 \to B_1$ mapping $\alpha_i, \beta_i, \partial X_j$ to $\alpha_i', \beta_i', \partial Y_j$, respectively, and agreeing with f_ϕ on the punctures of A_1. By construction, the triple (A_1, B_1, f_1) satisfies conditions (1)-(4).

Induction step: Suppose we have A_k, B_k and f_k satisfying conditions (1)-(4).

If k is even, let $K \geq k+1$ be such that $A_k \subsetneq \Sigma_K$ and set $A_{k+1} = \Sigma_K$.

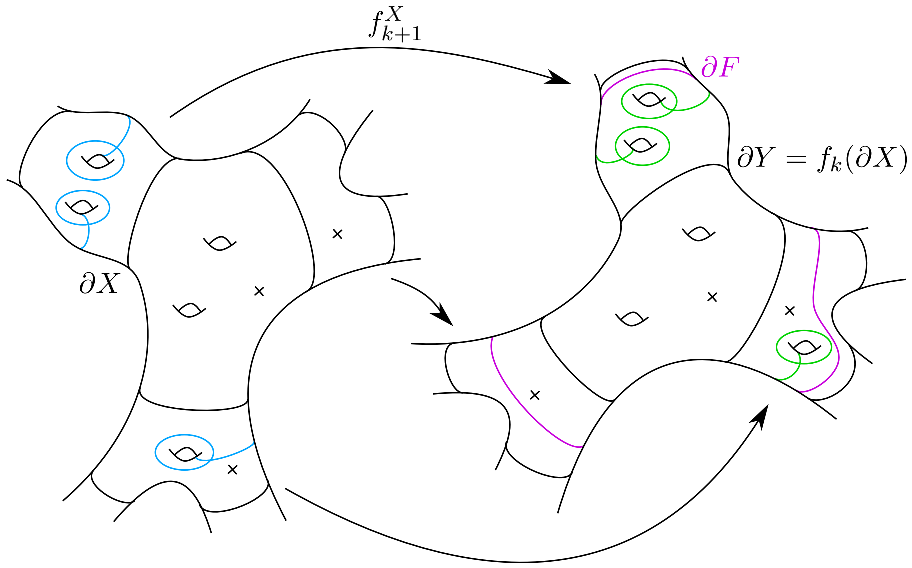

FIGURE 8. From k to $k+1$ in the proof of Theorem 2

Let X be a complementary component of A_k, and let Y be the complementary component of B_k bounded by $f_k(\partial X)$. Let q be the genus of $X \cap A_{k+1}$. Choose curves $\alpha_1, \beta_1, \dots, \alpha_q, \beta_q$ in $X \cap A_{k+1}$ with the standard symplectic intersection pattern.

By condition (4), we can realize the classes $\phi([\alpha_1]), \dots, \phi([\beta_q])$ by curves $\alpha_1', \dots, \beta_q'$ in Y with the standard intersection pattern. Choose a separating curve in Y bounding a surface F of genus q containing $\alpha_1', \beta_1', \dots, \alpha_q', \beta_q'$ and the images of the punctures of $X \cap A_{k+1}$ under f_ϕ.

Let $X_1, \dots, X_r$ be the flare surfaces X_i which are the components of $X \smallsetminus A_{k+1}$.

Arguing as in the base case, we can find disjoint flare surfaces Y_i in Y so that $\phi(\mathrm{H}_1(X_i; \mathbb{Z})) = \mathrm{H}_1(Y_i; \mathbb{Z})$ for all i. The boundaries ∂Y_i, together with ∂Y, cut off a compact subsurface $K_X \subset Y$ homeomorphic to $X \cap A_{k+1}$.

We can therefore choose a homeomorphisms $f_X^{k+1} \colon X \cap A_{k+1} \to K_X$ sending $\alpha_1, \beta_1, \dots, \alpha_q, \beta_q$ to $\alpha_1', \beta_1', \dots, \alpha_q', \beta_q'$, agreeing with f_k on ∂X and with f_ϕ on the set of punctures of $X \cap A_{k+1}$, and sending ∂X_i to ∂Y_i for all i.

Since all complementary flare surfaces X of A_k are disjoint, we can repeat this process independently on all of them, obtaining sets K_X and maps f_X^{k+1}. Now let B_{k+1} be the union of B_k with all K_X, that is,

$$B_{k+1} = B_k \cup \left(\bigcup_{X \in \pi_0(S \smallsetminus A_k)} K_X \right).$$

We form f_{k+1} by gluing the maps f_X^{k+1} to f_k:

$$f_{k+1} = f_k \cup \left(\bigcup f_X^{k+1} \right),$$

which is possible since f_k, f_X^{k+1} have pairwise disjoint supports. From the construction it immediately follows that f_{k+1} is a homeomorphism between A_{k+1} and B_{k+1} which has the desired properties.

When k is odd we proceed similarly, but we first define B_{k+1} and the curves there and then use ϕ^{-1} to get curves outside A_k and hence A_{k+1}. □

Appendix A. (Co)Homology Classes, Curves and Arcs

In this appendix we discuss and prove various results describing relations amongst simple curves, simple arcs, and (co)homology classes. In the case of finite-type surfaces, most of these results are well-known; we collect here extensions to the infinite-type setting, as well as some new characterizations.

Let us recall the notation we will use here. We say that a homology class $x \in \mathrm{H}_1(S, \mathbb{Z})$ is *realized by a simple closed curve* if there is a simple closed curve γ so that $[\gamma] = x$. We also say that x is a *simple (non-)isotropic* if x is realized by a simple closed curve and x is (non-)isotropic.

In the case of finite-type surfaces, the characterization of simple (non-)isotropics was done by Meeks and Patrusky [MP78].

Their answer requires the following construction, which is also useful in the study of infinite-type surfaces. Given a surface S, denote by $\hat{S}$ the surface obtained by filling in the planar ends of S, and gluing disks to all boundary components. Note that $\hat{S}$ is compact if S has finite genus.

Let $i\colon S \to \hat{S}$ be the natural inclusion and i_* the corresponding map induced on first homology. Observe that $\ker(i_*)$ is isotropic with respect to the algebraic intersection pairing $\hat{\imath}$ on $\mathrm{H}_1(S;\mathbb{Z})$, and therefore we have

$$\hat{\imath}(i_*x, i_*y) = \hat{\imath}(x, y) \quad \forall x, y \in \mathrm{H}_1(S;\mathbb{Z}).$$

In our language, we can now state the characterisation of simple (non-)isotropics for finite-type surfaces as follows.

Theorem A.1 ([MP78, Theorem 1]). *Let S be a finite-type surface with $n+1$ punctures and let $\gamma_1, \dots \gamma_{n+1}$ be disjoint curves surrounding the punctures, oriented so that the puncture is to the right. Let $x \in \mathrm{H}_1(S;\mathbb{Z})$ be a non-zero class.*

i) x is a simple non-isotropic if and only if $i_*x \in \mathrm{H}_1(\hat{S};\mathbb{Z})$ is a non-zero primitive class.
ii) x is a simple isotropic if and only if $x = \pm\sum_{i=1}^{n} \varepsilon_i[\gamma_i]$, where $\varepsilon_i \in \{0, 1\}$.

Observe in particular that the homology class of a separating simple closed curve δ is completely determined by the set of punctures lying to the left of δ, which is a special case of Lemma 2.3.

In the subsequent parts of this section, we will develop analogous characterizations of simple (non-)isotropics for infinite-type surfaces. The characterization of simple isotropics involves (algebraic) intersections with simple arcs joining two ends, and so we also characterize these in the final subsection.

A.1. Geometric homology bases.

In the case of surfaces of finite type, there are standard bases for homology that are considered; in particular, those given by one curve for all but one puncture and a geometric symplectic basis for the compactified surface. We want to describe standard bases for infinite-type surfaces as well.

A *geometric homology basis* for a surface S is a basis of homology $\{[\alpha_i], [\beta_i]\}_{i\in I} \cup \{[\gamma_j]\}_{j\in J}$, where the $\alpha_i, \beta_i, \gamma_j$ are all simple closed curves and such that

- $\{[\gamma_j]\}_{j\in J}$ is a basis for the isotropic subspace of $\mathrm{H}_1(S;\mathbb{Z})$,
- $i(\alpha_j,\alpha_k) = i(\beta_j,\beta_k) = 0$ for all $j,k \in I$,
- $i(\alpha_j,\beta_k) = \delta_{jk}$ for all $j,k \in I$, and
- for any compact subset K of S, only finitely many curves in the basis intersect K.

Lemma A.2. *For any surface S there exists a geometric homology basis.*

Proof. Consider an exhaustion of S by compact subsurfaces $\Sigma_1 \subset \Sigma_2 \subset \dots$ whose boundary curves are all separating and are allowed to be peripheral. Construct by induction a geometric symplectic basis for Σ_n with boundary capped off, by choosing such a basis for Σ_1 and extending the basis for Σ_n to a basis for Σ_{n+1}. This gives the $\{\alpha_i,\beta_i\}_{i\in I}$ with the required intersection numbers. To get the basis for the isotropic part, consider the set $\{\delta_l\}_{l\in L}$ of all boundary components of all Σ_n and let $\{\gamma_j\}_{j\in J}$ be a maximal independent subset. The space generated by $\{\gamma_j\}_{j\in J}$ is the same as the space generated by $\{\delta_l\}_{l\in L}$ and this is the isotropic subspace of $\mathrm{H}_1(S;\mathbb{Z})$. Furthermore, by construction, all γ_j are pairwise disjoint and disjoint from the other curves. As any compact set is contained in some Σ_n, for n big enough, and each subsurface contains only finitely many curves representing basis elements, we get the third property of a geometric basis as well. □

In the case of the Loch Ness monster surface, there is no isotropic subspace of $\mathrm{H}_1(S;\mathbb{Z})$ of homology and a geometric homology basis is described in Figure 9.

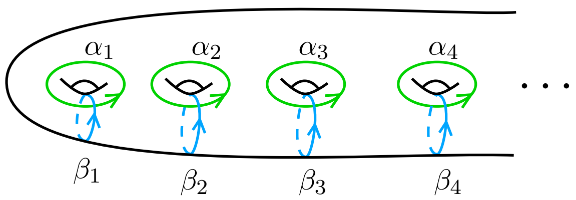

FIGURE 9. A geometric basis for the Loch Ness monster surface

Note that the same homology basis could be realized by two different sets of curves, one of which gives a geometric homology basis and the other of which does not. An example is given in Figure 10. The curves represent the same basis as the curves in Figure 9, but they are not a geometric homology basis: there is a compact subsurface intersecting all of the β_i'.

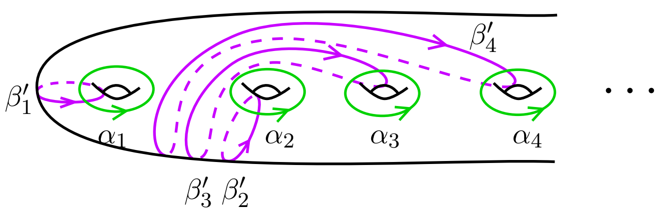

FIGURE 10. Curves representing the same basis as the curves in Figure 9

Moreover, two geometric symplectic bases do not need to be in the same mapping class group orbit, as the example in Figure 11 shows.

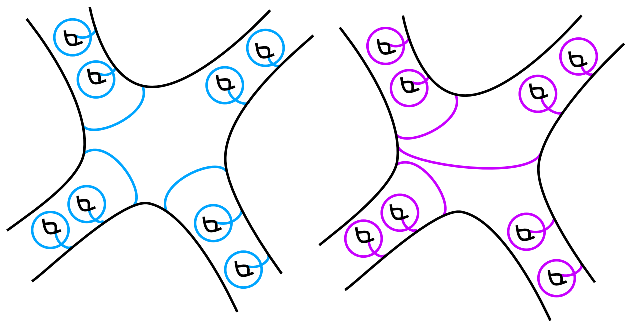

FIGURE 11. Two geometric symplectic bases that are not in the same mapping class group orbit.

We will also consider the first cohomology group with integer coefficients $\mathrm{H}^1(S;\mathbb{Z})$ of a surface S. We will often use the identification

$$\mathrm{H}^1(S;\mathbb{Z}) \simeq \mathrm{Hom}(\mathrm{H}_1(S;\mathbb{Z});\mathbb{Z})$$

without explicit mention.

A.2. **Non-separating curves.** We begin with a simple criterion to detect classes realizable by non-separating simple closed curves, which is likely well known.

Lemma A.3 (Lemma 2.2)**.** *Let S be any surface and $x \in \mathrm{H}_1(S;\mathbb{Z})$. Then x is a simple non-isotropic if and only if there exists $y \in \mathrm{H}_1(S;\mathbb{Z})$ such that $\hat{\imath}(x,y) = 1$.*

Proof. First observe that for any non-separating simple closed curve α there exists a curve β that intersects it geometrically once, and hence $\hat{\imath}([\alpha], \pm[\beta]) = 1$. This shows one direction of the lemma.

For the other direction, we begin with the case where S is finite type. Since $\hat{\imath}(i_*x, i_*y) = \hat{\imath}(x,y) = 1$, the homology class $i_*x \in \mathrm{H}_1(\hat{S};\mathbb{Z})$ is primitive. By Theorem A.1, this implies that x is realized by a non-separating simple closed curve.

In the case of a general S, there is a finite-type subsurface $F \subset S$ so that x, y can be realized by loops on F. Applying the previous case to F then shows that x can be realized by a simple closed curve on F, hence S. □

Lemma A.4. *Suppose that $\{[\alpha_i],[\beta_i]\}_{i\in I} \cup \{[\gamma_j]\}_{j\in J}$ is a geometric basis for homology, where $\{[\gamma_j]\}_{j\in J}$ is a basis for the isotropic subspace of $\mathrm{H}_1(S;\mathbb{Z})$. Then any primitive element in the span of $\{[\alpha_i],[\beta_i]\}_{i\in I}$ can be realized by a non-separating simple closed curve.*

Proof. Arguing as in the proof of Lemma 2.2, by passing to a suitable subsurface it suffices to show this for surfaces of finite type. Now, the lemma follows from Theorem A.1, since i_* induces an isomorphism between the span of the $\{[\alpha_i],[\beta_i]\}_{i\in I}$ and $\mathrm{H}_1(\hat{S};\mathbb{Z})$. □

Instead of requiring the existence of a class that intersects correctly, we can also characterize classes realized by non-separating simple closed curves by an extension property.

Lemma A.5. *Let S be any surface and $x \in \mathrm{H}_1(S;\mathbb{Z})$. Then x is a simple non-isotropic if and only if x is not isotropic and there is a basis B of $\mathrm{H}_1(S;\mathbb{Z})$ so that*

i) $x \in B$, and
ii) B contains a basis for the isotropic subspace of $\mathrm{H}_1(S;\mathbb{Z})$.

Proof. One direction follows since any non-separating simple closed curve can be extended to a standard basis for homology.

For the reverse direction, suppose x is not isotropic and can be extended to a basis $\{x, x_k, y_l\}$ where $\{y_l\}$ is a basis for the isotropic subspace.

Choose a geometric basis for homology $\{[\alpha_i], [\beta_i]\}_{i\in I} \cup \{[\gamma_j]\}_{j\in J}$, where $\{[\gamma_j]\}_{j\in J}$ is a basis for the isotropic subspace of $\mathrm{H}_1(S;\mathbb{Z})$. Write x in this geometric basis as the sum

$$x = X + Y$$

where X is a linear combination of $\{[\alpha_i], [\beta_i]\}_{i\in I}$ and Y is a linear combination of $\{[\gamma_j]\}_{j\in J}$.

Since x is not isotropic we have $X \neq 0$. We claim that X is in fact primitive. Assuming this claim, by combining Lemma A.4 and Lemma 2.2, we can find a homology class w so that $1 = \hat{\imath}(w, X)$. We then have $1 = \hat{\imath}(w, X) = \hat{\imath}(w, x)$, and we are done by Lemma 2.2.

To prove the claim, suppose $X = nX_0$ for some $n \in \mathbb{Z}$. Since Y is isotropic we have

$$x - nX_0 = Y = \sum n_j y_j.$$

Let $m \in \mathbb{Z}$ be the coefficient of x when writing X_0 in the basis $\{x, x_k, y_l\}$. Then by looking at the coefficient of x in the previous equation, we obtain

$$1 - nm = 0$$

which implies that $nm = 1$, i.e. $n = \pm 1$. □

Remark A.6. In Lemma A.5 is is actually necessary to require that the basis B contains a basis for the isotropic subspace. Namely, let S be a twice-punctured torus with standard geometric basis given by two non-separating curves α and β intersecting once and one of the two boundary curves denoted by γ. Consider the class $x = [\gamma] + 2[\alpha]$. It cannot be realized by a non-separating curve by Theorem A.1, but it is non-isotropic and $[\alpha], [\beta], x$ is a basis of homology.

A.3. **Separating Curves.** To characterize simple isotropics, we will use intersections with arcs. Recall that we assume arcs to be properly embedded, but allow them to be non-compact.

Lemma A.7. *Let S be any surface and $x \in \mathrm{H}_1(S;\mathbb{Z})$. Then x is a simple isotropic if and only if x is isotropic and $|\,\hat{\imath}(x, a)| \leq 1$ for every simple arc.*

Proof. One direction is easy: if γ is a simple separating curve, then $\hat{\imath}(\gamma, a)$ is ± 1 if γ separates the ends that a joins; otherwise the intersection is zero as any two successive intersections must have different signs.

We begin by showing the other direction in the case of a finite-type surface S with $n+1$ punctures. Let $\gamma_1, \ldots, \gamma_{n+1}$ be loops, each surrounding a puncture and oriented so that the puncture is to the left. Then any collection of n elements of the set $\{\gamma_1, \ldots, \gamma_{n+1}\}$ is a basis for the isotropic subspace of $\mathrm{H}_1(S, \mathbb{Z})$. Let x be an isotropic homology class and suppose that $|\,\hat{\imath}(x, a)| \leq 1$ for every arc joining punctures. As x is isotropic, we have

$$x = \sum_{i=1}^{n} c_i [\gamma_i].$$

Consider an arc α from γ_j to γ_{n+1}; then

$$1 \geq |\,\hat{\imath}(x, \alpha)| = \left| \sum_{i=1}^{n} c_i \, \hat{\imath}([\gamma_i], \alpha) \right| = |c_j|$$

so each coefficient has absolute value at most one. If there were two indices such that $c_j = 1$ and $c_k = -1$, then any arc connecting γ_j and γ_k would have algebraic intersection number ± 2 with x. Hence all non-zero coefficients have the same sign and using Theorem A.1 we deduce that x can be realized by a simple closed curve.

Finally, suppose that S is of infinite type, and that x is as in the lemma. Choose a finite type surface $F \subset S$ which contains a loop homologous to x and so that every boundary curve of F is separating in S. By the latter property, any simple arc $a_0 \subset F$ can be extended to a simple arc a in S so that $a \cap F = a_0$. Hence, we can apply the finite-type case to x and F, and conclude that x is a simple isotropic in F, and hence in S. □

A.4. **Arcs.** Given an arc α joining two ends, we have an associated integral cohomology class $\hat{\imath}(\alpha, \cdot)$. The goal of this section is to characterize which cohomology classes arise this way.

Given $f \in \mathrm{H}^1(S, \mathbb{Z})$, we say that:

- f *has support in* $e \in \mathrm{Ends}(S)$ if for all $V \in \mathcal{F}_e$ there is $x \in V$ such that $f(x) \neq 0$;
- the *support* $\mathrm{supp}(f)$ of f is the set of ends in which f has support;
- f is *arclike* in $e \in \mathrm{Ends}(S)$ if for all $V \in \mathcal{F}_e$ there is $x \in V$ such that x isotropic and $f(x) = 1$.

The goal of the next set of lemmas is to prove the following:

Proposition A.8. *Let $f \in \mathrm{H}^1(S; \mathbb{Z})$ be such that*

(1) the support of f is $\{e_1, e_2\}$, for $e_1 \neq e_2 \in \mathrm{Ends}(S)$, and
(2) f is arclike in e_1 and e_2.

Then $f = \hat{\imath}(\alpha, \cdot)$ for a simple arc α connecting e_1 and e_2.

We start with a lemma concerning the support of cohomology classes:

Lemma A.9. *Let $f \in \mathrm{H}^1(S; \mathbb{Z})$. Suppose Σ is a star surface with a boundary component γ such that $f([\gamma]) \neq 0$. Then $\mathrm{supp}(f) \cap \mathcal{R}(\gamma) \neq \emptyset$.*

Proof. Fix an exhaustion by star surfaces F_n.

Denote by X_γ the component of $S \setminus \gamma$ to the right of γ. Choose N so that γ is contained in the interior of F_N. Then $\Sigma_1 = F_N \cap X_\gamma$ is a star surface with $\Sigma \cap \Sigma_1 = \gamma$. The sum of the classes of the boundary components and of the curves surrounding one puncture of Σ_1 is zero in homology. So there must be one such curve γ_1, different from γ, which satisfies $f([\gamma_1]) \neq 0$.

If $\mathcal{R}(\gamma_1)$ is a single puncture, we are done. Otherwise we can repeat the process with Σ_1 and γ_1 instead of Σ and γ.

If we find a Σ_n and a curve $\gamma_n \subset \Sigma_n$ with $\mathcal{R}(\gamma_n) = \{e\}$, we conclude as above that $e \in \mathrm{supp}(f) \cap \mathcal{R}(\gamma_n) \subset \mathrm{supp}(f) \cap \mathcal{R}(\gamma)$.

Otherwise, we get an infinite sequence of surfaces and a sequence curves γ_n going to infinity, and hence accumulating to an end e in $\mathcal{R}(\gamma)$, on which f is non-zero. Thus $e \in \mathrm{supp}(f)$. □

An easy consequence of the previous lemma is the following:

Corollary A.10. *If $f \in \mathrm{H}^1(S; \mathbb{Z})$ has $\mathrm{supp}(f) = \{e_1, e_2\}$, then for any star surface Σ there are at most two boundary curves γ_1 and γ_2 on which f is non-zero.* □

We will also need a characterization of intersection with arcs in the case of finite-type surfaces. It relies on the following lemma.

Lemma A.11. *Suppose that Σ is a closed surface of finite type and that α is a simple closed curve. If f is a cohomology class on Σ with $f([\alpha]) = 1$, then there is a simple closed curve β so that*

(1) $f(x) = \hat{\iota}(x, [\beta])$ for all $x \in \mathrm{H}_1(\Sigma; \mathbb{Z})$, and
(2) α and β intersect in a single point.

Proof. The algebraic intersection form $\hat{\iota}$ is a non-degenerate symplectic form in this case, so there exists a homology class b with $f(x) = \hat{\iota}(x, b)$ for all $x \in \mathrm{H}_1(\Sigma; \mathbb{Z})$. Since $f(\alpha) = 1$, the class b is primitive, and can thus (by Theorem A.1) be realized by a simple closed curve β, showing (1). The fact that β can be chosen to intersect α in a single point can be shown as in [FM12, Theorem 6.4]. $\square$

We can now prove the finite-type version of Proposition A.8.

Lemma A.12. *Let Σ be a compact surface and $\gamma_1, \dots, \gamma_n$ be its boundary components. Suppose $f \in \mathrm{H}^1(\Sigma; \mathbb{Z})$ is such that there are two indices i_1, i_2 for which*

$$f([\gamma_{i_1}]) = -f([\gamma_{i_2}]) = 1,$$

and

$$f([\gamma_j]) = 0, \quad \forall j \neq i_1, i_2.$$

Then $f = \hat{\iota}(\alpha, \cdot)$ for a simple arc α connecting γ_{i_1} and γ_{i_2}.

Proof. Let F be the surface obtained from Σ by gluing γ_{i_1} and γ_{i_2}. We can describe the homology of F as follows:

$$\mathrm{H}_1(F; \mathbb{Z}) = \mathbb{Z} \oplus \mathrm{H}_1(\Sigma; \mathbb{Z}) / \langle [\gamma_{i_1}] + [\gamma_{i_2}] \rangle,$$

where the obvious map from Σ to F corresponds to the quotient map

$$\mathrm{H}_1(\Sigma; \mathbb{Z}) \to \mathrm{H}_1(\Sigma; \mathbb{Z}) / \langle [\gamma_{i_1}] + [\gamma_{i_2}] \rangle.$$

By the first assumption, the cohomology class f on Σ descends to a form on $\mathrm{H}_1(\Sigma; \mathbb{Z}) / \langle [\gamma_{i_1}] + [\gamma_{i_2}] \rangle$, and we extend this to a form f_F on $\mathrm{H}_1(F; \mathbb{Z})$ by letting it be 0 on the summand $\mathbb{Z}$.

Let S be the closed surface obtained by gluing a disc D_j to the boundary component γ_j of F for each $j \neq i_1, i_2$. Observe that

$$\mathrm{H}_1(S; \mathbb{Z}) = \mathrm{H}_1(F; \mathbb{Z}) / \langle [\gamma_j], j \neq i_1, i_2 \rangle$$

By the second assumption on f, the class f_F descends to a cohomology class f_S on S. Observe that $f_S(\gamma_{i_1}) = 1$, and hence we can apply Lemma A.11 to find a curve $\bar{\beta}$ which intersects γ_{i_1} in a single point, and which satisfies

$$\iota(x, \bar{\beta}) = f_S(x), \quad \forall x \in \mathrm{H}_1(S; \mathbb{Z}).$$

We may assume that $\bar{\beta}$ is disjoint from all the discs D_j, and therefore defines a curve $\beta \subset F$, which now has

$$\iota(x, \beta) = f_F(x), \quad \forall x \in \mathrm{H}_1(F; \mathbb{Z}).$$

and still intersects γ_{i_1} in a single point.

The preimage of β on Σ is then the desired arc. $\square$

Proof of Proposition A.8. In this proof, we will allow subsurfaces to have boundary components homotopic to punctures and to be annuli with (both) boundary curves homotopic to a puncture.

Consider an isotropic class x such that $f(x) = 1$ and let Σ_0 be a compact star surface such that $x \in \mathrm{H}_1(\Sigma_0, \mathbb{Z})$. Let $\gamma_0^1, \dots \gamma_0^m$ be the boundary components of Σ_0.

Choose a compact exhaustion F_n such that $\Sigma_0 = F_0$ and no two boundary components are homotopic, unless they are homotopic to a single puncture.

By Corollary A.10, there are at most two boundary components of Σ_0, say γ_0^1 and γ_0^2, on which f is not zero. Since the sum of the boundary components of Σ_0 is zero in homology, $f(\gamma_0^1) = -f(\gamma_0^2)$. Furthermore $\{[\gamma_0^i] \,|\, i = 1, \dots, m-1\}$ is a basis of the isotropic part, hence there are $c_i \in \mathbb{Z}$ with

$$x = \sum_{i=1}^{m-1} c_i[\gamma_0^i].$$

Applying f we get

$$1 = c_1 f(\gamma_0^1) + c_2 f(\gamma_0^2) = (c_1 - c_2) f(\gamma_0^1)$$

which implies that $f(\gamma_0^1) = \pm 1$ and $f(\gamma_0^2) = \mp 1$. Note that by Lemma A.9 (and up to changing the labels of the end in the support of f), we have $e_i \in \mathcal{L}(\gamma_0^i)$. As f has support only in two ends, up to enlarging Σ_0 we can assume that $f \neq 0$ only on Σ_0 union the two connected components of $S \smallsetminus \Sigma_0$ which have γ_0^1 or γ_0^2 in their boundary, i.e. those containing e_1 or e_2. Now apply Lemma A.12 to get that $f|_{\Sigma_0} = \hat{\imath}(\alpha_0, \cdot)$ for some simple arc $\alpha_0 \subset \Sigma_0$ connecting γ_0^1 to γ_0^2. Denote by X_0^i the flare surface to the right of γ_0^i.

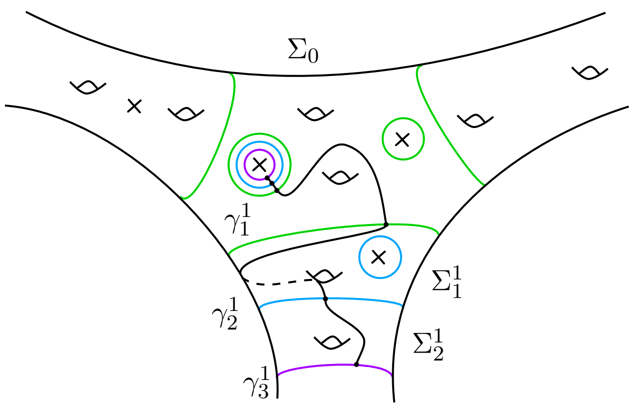

FIGURE 12. An example of the sequence of subsurfaces constructed in the proof of Proposition A.8

Consider $\Sigma_1^i = F_1 \cap X_0^i$ for $i = 1, 2$ and if necessary enlarge them so that f is zero on all components of $S \smallsetminus (\Sigma_0 \cup \Sigma_1^1 \cup \Sigma_1^2)$ not containing e_1 or e_2. Repeat the argument to get that f is given by intersection in arcs α_1^i in these subsurfaces. Slide the endpoints on the boundary components so that the arcs can be glued to α_0; the resulting arc defines f on the union of the three star surfaces.

Repeat the process to get that f is represented by $\hat{\imath}(\alpha, \cdot)$ on a (countably infinite) union of star surfaces, where α is a simple arc joining e_1 and e_2. By construction, f is zero outside of the union, hence $f = \hat{\imath}(\alpha, \cdot) \in \mathrm{H}^1(S; \mathbb{Z})$. □

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