HOMOLOGICAL APPROXIMATIONS TO THE HANDLEBODY GROUP

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Abstract. We characterise which elements of the mapping class group of a surface extend to a handlebody or compression body using only the action on the homology of finite covers. We also begin to investigate which subspaces in the homology of finite covers are defined by meridians in handlebodies.

1. Introduction

Consider a closed surface Σ of genus $g \geq 2$. The mapping class group $\mathrm{Mcg}(\Sigma)$ of Σ is the group of diffeomorphisms of Σ up to isotopy. One basic tool in the study of $\mathrm{Mcg}(\Sigma)$ is the *homology representation*, given by the action of diffeomorphisms on the homology of the surface. Explicitly, there is a short exact sequence

$$1 \to \mathcal{I}_g \to \mathrm{Mcg}(\Sigma) \to \mathrm{Sp}(2g, \mathbb{Z}) \to 1$$

where $\mathcal{I}_g$ is the *Torelli group*. While this homology representation is useful, much information is lost – the Torelli group already exhibits all of the complications encountered when studying the mapping class group.

In recent years, interest has surfaced in *virtual homology representations* of the mapping class group, that is: representations of finite index subgroups of $\mathrm{Mcg}(\Sigma)$, acting on the homology of finite covers of the surface Σ. These carry much finer information than the standard homology representation. To name a few examplary results: it is known that every element in $\mathrm{Mcg}(\Sigma)$ will act nontrivially on the homology of some cover [Kob]; such representations can used to construct arithmetic quotients of mapping class groups [GLLM], and their image is closely related to the Ivanov conjecture [PW].

In this article, we begin a study of *compression* of elements in $\mathrm{Mcg}(\Sigma)$ from this point of view. To phrase the main result, fix an identification of Σ with the boundary of a 3–dimensional *handlebody* V. The *handlebody subgroup* of $\mathrm{Mcg}(\Sigma)$ is the group $\mathcal{H}$ formed by those diffeomorphisms which extend to V.

If $\phi \in \mathcal{H}$, then the induced automorphism ϕ_* of $H_1(\Sigma; \mathbb{Z})$ preserves the Lagrangian submodule

$$L = \ker\left(H_1(\Sigma; \mathbb{Z}) \to H_1(V; \mathbb{Z})\right) < H_1(\Sigma; \mathbb{Z}).$$

However, it is known that there are elements of the Torelli group (or in fact any term of the Johnson filtration) which are not conjugate into $\mathcal{H}$ [Jor]. In other words, homology of the surface is unable to detect compression.

Our first main result shows that the situation is very different if one is willing to consider the homology of finite covers. Intuitively, the result says that if a mapping class group element acts on the homology of enough finite covers of a surface like a handlebody group element, then it is a handlebody group element.

To make this precise, we define *homological approximations* $\Gamma^{(n)}$ to the handlebody group. Namely, inductively define $V^{(n)}$ as iterated mod-2-homology covers of the handlebody $V = V^{(0)}$. That is, $V^{(n+1)}$ is the cover of $V^{(n)}$ defined by

$$\pi_1(V^{(n)}) \to H_1(V^{(n)}; \mathbb{Z}/2\mathbb{Z}).$$

The choice of "mod-2" is not relevant and is just chosen for concreteness here. Let $\Sigma^{(n)} = \partial V^{(n)}$ be the boundary. For each n, define the Lagrangian submodule

$$L^{(n)} = \ker\left(H_1(\Sigma^{(n)}; \mathbb{Z}) \to H_1(V^{(n)}; \mathbb{Z})\right).$$

We let $\Gamma^{(n)} < \mathrm{Mcg}(\Sigma)$ to be the subgroup of all ϕ which lift to $\Sigma^{(n)}$, and so that a lift preserves $L^{(n)}$. If $\phi \in \Gamma^{(n-1)}$ then it will automatically lift to $\Sigma^{(n)}$, and so being contained in $\Gamma^{(n)}$ really is a purely homological criterion.

Theorem 1. *With terminology as above,*

$$\bigcap_{n=0}^{\infty} \Gamma^{(n)} = \mathcal{H}.$$

For every n, the index of $\mathcal{H}$ in $\Gamma^{(n)}$ is infinite.

For a given $\phi \in \mathrm{Mcg}(\Sigma)$, the number $n = n(\phi)$ so that $\phi \in \mathcal{H}$ if and only if $\phi \in \Gamma^{(n)}$ can be computed from geometric properties of the mapping class ϕ (compare Corollary 3.2).

Shifting perspective for a moment, Theorem 1 has a consequence for the structure of $\mathcal{H}$ as a subgroup of the mapping class group. Recall that a subgroup H of a group G is *seperable*, if H is equal to the intersection of all finite index subgroups of G which contain H. We have

Corollary 1. *The handlebody group $\mathcal{H}$ is seperable in the mapping class group* $\mathrm{Mcg}(\Sigma)$.

This was proved before in [LM], but our methods are different: the treatment in [LM] concludes seperability abstractly using the action of the mapping class group on suitable representation varieties, whereas in our approach the finite quotients which certify seperability are explicit and natural from the perspective of surface topology.

Our methods also show a version of Theorem 1 and Corollary 1 for the *compression body group*, which is the subgroup of those mapping classes which extend to some fixed compression body.

Theorem 1 is a consequence of being able to detect which loops $\gamma \in \pi_1(\Sigma)$ represent the trivial element in $\pi_1(V)$ via the homology of finite covers. This is facilitated by the following construction. Given a loop γ, an *elevation* of γ to $\Sigma^{(n)}$ is a lift of a minimal power of γ. We denote by $[\gamma]_n \in H_1(\Sigma^{(n)}; \mathbb{Z})$ the homology class defined by an elevation. Consider the set of those loops whose elevations define classes in the Lagrangians $L^{(n)}$:

$$\Pi(L) = \{\gamma \in \pi_1(\Sigma) \mid [\gamma]_n \in L^{(n)} \text{ for all } n\}$$

The main ingredient for Theorem 1 is that if the $L^{(n)}$ are defined by the identification of Σ with the boundary ∂V of a handlebody, then as we will show in Theorem 3.1:

$$\Pi(L) = \ker(\pi_1(\Sigma) \to \pi_1(V))$$

One can think of this as a geometric version of residual finiteness of free groups. In Proposition 3.5 we develop a version for compression bodies.

Theorem 1 describes homologically when a mapping class $\phi \in \mathcal{H}$. From a topological perspective, it would be desirable to determine if ϕ is conjugate into $\mathcal{H}$ – this exactly corresponds to asking if ϕ extends to the handlebody under *some* identification of Σ with the boundary of a handlebody. Such *compressability* can have topological consequences. To name one prominent example, the main result of [CG] shows that a fibered knot in S^3 is homotopically ribbon if and only if its monodromy compresses. Compressability of mapping classes has been studied in [Bon] in the language of relative cobordism groups, and in [CL] from an algorithmic perspective.

In the context of Theorem 1, one should think of the different identifications of Σ with ∂V as different *towers* of Lagrangian summands $L^{(n)}$ of homology groups of finite covers. To describe compressability using homology of finite covers, the central issue is to determine which of these towers are *geometric*: i.e. actually arise from an identification of Σ with the boundary of a handlebody.

This turns out to be an intricate question and we will give two answers of very different flavours. On the one hand, one can use the virtual homology representations of the mapping class group to describe which towers are geometric (compare Section 5.2). As this action is not very well-understood for complicated covers, this description is not explicit. It can be used to show that the possible $L^{(0)}, L^{(1)}$ which appear in geometric towers can be described, and satisfy very few constraints (compare Corollary 5.12 and the discussion preceding it).

There is a different perspective which yields a more explicit answer. Namely, if we are given any tower $(\Sigma^{(n)})$ defined by Lagrangian direct summands $L^{(n)}$ of $H_1(\Sigma^{(n)}; \mathbb{Z})$ (see Section 2.3 for precise definitions), we can define a set $\Pi(L)$ as above. We will show (as Theorem 5.5) that towers are geometric, as soon as $\Pi(L)$ is "big enough":

Theorem 2. *The tower L arises from the identification of Σ with ∂V if and only if the image of $\Pi(L)$ in $H_1(\Sigma; \mathbb{Z})$ generates $L^{(0)}$.*

In Section 5.1 and 6 we will explain that checking if $\Pi(T)$ is nontrivial eventually relies on understanding which classes in the homology of a cover are obtained as elevations of simple loops in $\pi_1(\Sigma)$. This is known to be a hard problem, which has received a lot of attention recently, and lies at the heart of understanding virtual homology representations of the mapping class groups. Compare e.g. [FH1] and the references therein for details.

The article is structured as follows. In Section 2 we collect some basic results on handlebodies and then introduce our basic object of study: *homological covering towers*. Section 3 contains the proof of the first claim of Theorem 1. Section 4.1 is concerned with proving that any finite cover cannot detect the handlebody group (the second part of Theorem 1), and thus also cannot distinguish meridians. In

Section 5 we prove Theorem 2, and describe the connection of realisability with the action of mapping class group. In particular, we discuss a genus 2 example in detail, which may be of independent interest for readers interested in the action of the mapping class group on covers. Finally, in Section 6 we relate the results to the notion of *prohomology*, studied by Koberda [Kob] and Boggi [Bog].

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2. Preliminaries

In this section, we describe the central objects of this article and set up some basic notation.

2.1. **Basic notation on covers.**

We begin by recalling some basic notation on covers, which is not completely standard.

Suppose that $p : \Sigma' \to \Sigma$ is a finite cover. In this setting, we always identify the fundamental group of Σ' with a subgroup of $\pi_1(\Sigma)$. The cover is *regular*, if Σ is the quotient of Σ' by the deck group G of Σ'. Equivalently, if $\pi_1(\Sigma') < \pi_1(\Sigma)$ is normal with quotient G. We say that a regular cover Σ' has a group-theoretic property (like Abelian, nilpotent etc.) if the deck group G has this property.

If γ is a loop on Σ, then we say that γ *lifts (with degree* 1*)* if $\gamma \in \pi_1(\Sigma')$. Equivalently, the loop γ lifts to a closed loop in Σ'.

For any loop γ, the *degree of lifting* $k(\gamma)$ is the smallest natural number so that $\gamma^{k(\gamma)} \in \pi_1(\Sigma')$. The lifts of $\gamma^{k(\gamma)}$ are called the *elevations of* γ.

If γ is simple, then the elevations are the connected components of the preimage $p^{-1}(\gamma)$ of γ under the covering map.

2.2. **Handlebodies and Compression Bodies.**

A *handlebody* V of genus g is the 3–manifold with boundary obtained from the 3–dimensional ball B^3 by attaching g one-handles. The boundary ∂V of V is a closed surface of genus g.

In this section we collect some basic, well-known results and terminology on handlebodies which will be used throughout.

A *meridian for* V is an essential, simple closed curve α on ∂V which is the boundary of a disk in V. A *cut system* for V is a collection $\alpha_1, \ldots, \alpha_g$ of disjoint meridians, so that $\partial V - (\alpha_1 \cup \cdots \cup \alpha_g)$ is connected.

The following two lemmas are well-known and easy.

Lemma 2.1. *If* $\alpha_1, \ldots, \alpha_g$ *is a cut system, then the homology classes* $[\alpha_i] \in H_1(\partial V; \mathbb{Z})$ *are a basis for the kernel*

$$\ker \left(H_1(\partial V; \mathbb{Z}) \to H_1(V; \mathbb{Z}) \right)$$

of the map induced by the inclusion $\partial V \subset V$*.*

This kernel is a direct summand of $H_1(\partial V; \mathbb{Z})$ *and Lagrangian for the algebraic intersection pairing on* $H_1(\partial V; \mathbb{Z})$*.*

Lemma 2.2. *Let $\alpha_1, \ldots, \alpha_g$ be a cut system for V. Suppose that $\gamma \in \pi_1(\partial V)$ is a loop, which up to free homotopy on ∂V is disjoint from $\alpha_1 \cup \cdots \cup \alpha_g$. Then γ is nullhomotopic in V.*

Proof. If $\alpha_1, \ldots, \alpha_g$ is a cut system, then there are disjoint disks $D_i, \partial D_i = \alpha_i$ so that $V - (D_1 \cup \cdots \cup D_g$ is a 3–ball. The result follows since a ball is contractible. □

A *system of disks* for a handlebody V is a collection of meridians each complementary component of which is planar. Alternatively, the disks bounded by the meridians cut the handlebody into a disjoint union of 3-balls. The following is also standard.

Lemma 2.3. *Suppose that $D = \{\alpha_1, \ldots, \alpha_k\}$ is a system of disks. Then there is a cut system $C \subset D$ for V.*

If i is fixed so that α_i is contained in the closure of two different complementary components of D, then C can be assumed not to contain α_i.

Proof. We induct on the number of complementary components of D. If this is one, then D is in fact a cut system. Otherwise, let $c \in D$ be a component which is contained in the closures of two distinct complementary components of D. Since gluing two planar surfaces along a boundary component yields a planar surface, $D \setminus \{c\}$ is still a system of disks, but with fewer complementary components.

The second claim is immediate, as the first inductive step can be taken to choose α_i. □

If M is a manifold, we denote by $\mathrm{Mcg}(M)$ the *mapping class group* of M, i.e. the group of homeomorphisms of M up to isotopy.

Lemma 2.4. *Let V be a handlebody. Suppose that $\phi \in \mathrm{Mcg}(\partial V)$. Then ϕ extends to a homeomorphism of V if and only if for each meridian α, the simple closed curve $\phi(\alpha)$ is also a meridian. In fact, it suffices that for some cut system C, the image $\phi(C)$ is also a cut system.*

Proof. If both C and $\phi(C)$ are cut systems, extend ϕ to a map which sends $\Sigma \cup \mathcal{D}(C)$ to $\Sigma \cup \mathcal{D}(\phi(C))$ where $\mathcal{D}(C)$ is a disjoint collection of disks bounded by C. Now the claim follows as every homeomorphism between spheres extends to one between balls, and the fact that $V - \mathcal{D}(C)$ is a ball for each cut system. □

If Σ is a surface, and we have fixed a homeomorphism $f : \Sigma \to \partial V$, then we call the subgroup $\mathcal{H} < \mathrm{Mcg}(\Sigma)$ of those classes which extend to homeomorphisms of V the *handlebody group*. This terminology is slightly misleading, as $\mathcal{H}$ depends on the identification f. Different choices lead to conjugate groups $\mathcal{H}$.

For us, a *compression body* C is the 3–manifold which is obtained from a surface Σ by attaching a number of 1–handles at disjoint curves, and then gluing in a ball at each sphere component of the boundary of the result. The surface Σ is called the *outer boundary* of the compression body. As for handlebodies, a *meridian* is a simple closed curve on the outer boundary which bounds a disk in the compression body. Compare [Bon] for details on compression bodies. We need the following structure lemma.

Lemma 2.5. *If C is a compression body, then there are disjoint simple closed curves $\delta_1, \dots, \delta_k$ on the outer boundary of C so that*

a) Each δ_i is separating on the outer boundary.
b) Each δ_i bounds a disk $D_i \subset C$, all of which are disjoint.
c) There is exactly one complementary component of $D_1, \dots, D_k$ which is a handlebody, possibly of genus 0.
d) Any other complementary components of $D_1, \dots, D_k$ are homeomorphic to a trivial interval bundle over some orientable surface.

The following is an immediate consequence of the previous lemma.

Lemma 2.6. *Suppose C, δ_i are as in Lemma 2.5 then the following holds. Denote by $\Sigma_0, \Sigma_1, \dots, \Sigma_k$ the complementary components of the δ_i, so that Σ_0 is the boundary of the handlebody. Then the map induced by the inclusion*

$$\bigoplus_{i=1}^{k} H_1(\Sigma_i;\mathbb{Z})/\langle[\delta_1], \dots, [\delta_k]\rangle \to H_1(C;\mathbb{Z})$$

is injective.

2.3. **Homological Covering Towers.** In this section we define the central object of this article.

Defintion 2.7. *Let Σ be a surface, and $q > 1$ a natural number. Suppose that*

$$L < H_1(\Sigma;\mathbb{Z})$$

is a $\mathbb{Z}$–submodule. The homology cover defined by (L, q) *is the cover $\Sigma' \to \Sigma$ defined by the surjective map*

$$\pi_1(\Sigma) \to H_1(\Sigma;\mathbb{Z}) \to H_1(\Sigma;\mathbb{Z}/q\mathbb{Z}) \to H_1(\Sigma;\mathbb{Z}/q\mathbb{Z})/L_q = G$$

where

$$L_q < H_1(\Sigma;\mathbb{Z}/q\mathbb{Z})$$

is the image of L under the natural reduction map.

Any homology cover is a regular, finite cover with an Abelian deck group G. The homology cover defined by $(\{0\}, q)$ coincides by definition with the usual mod–q homology cover of Σ.

It is well-known that a $\mathbb{Z}$–submodule L of $H_1(\Sigma;\mathbb{Z})$ is a direct summand of $H_1(\Sigma;\mathbb{Z})$ if and only if there are simple closed curves $\alpha_1, \dots, \alpha_k$ on Σ whose homology classes generate L (see e.g. [Put, Lemma A.3]). To simplify notation, we will simply say that L *is generated by the curves* $\alpha_1, \dots, \alpha_k$.

The following lemma gives a useful characterisation of homology covers and is basically a consequence of the fact that the intersection pairing is natural with respect to reduction mod q.

Lemma 2.8. *If L is generated by $\alpha_1, \dots, \alpha_k$, then the fundamental group $\pi_1(\Sigma') < \pi_1(\Sigma)$ of the (L, q) homology cover consists exactly of those loops which have algebraic intersection number 0 mod q with all α_i.*

Proof. If L is a direct summand of $H_1(\Sigma;\mathbb{Z})$, then there is a symplectic complement $L^\perp$ consisting of all homology classes which have algebraic intersection number 0 with all elements of L. Since the classes $[\alpha_1], \dots, [\alpha_k]$ generate L, the complement $L^\perp$ equivalently consists of all classes which have algebraic intersection 0 with all α_i.

By Lemma A.3 of [Put], there are curves α_i, β_i intersecting geometrically, so that a subset generate L, and the complement generate $L^\perp$. We thus have

$$H_1(\Sigma, \mathbb{Z}/q\mathbb{Z}) = L_q \oplus L_q^\perp$$

and indeed L_q consists of the curves which have algebraic intersection 0 mod q with every class in L. This shows the lemma. □

Defintion 2.9. *i) A* tower of covers T *is an infinite sequence*

$$\cdots \to \Sigma_n \to \Sigma_{n+1} \to \cdots \to \Sigma_1 \to \Sigma$$

of finite covers of Σ. *We call the* Σ_n *the* levels *of the tower.*

ii) We say that T *is* regular *if each* $\Sigma_n \to \Sigma$ *is a regular cover. In that case, we denote by* G_n *the deck group of the cover* $\Sigma_n \to \Sigma$.

iii) We say that T *is* homological *if for each* n*, the cover* $\Sigma_{n+1} \to \Sigma_n$ *is the* (L_n, q_n)*–homology cover for some* L_n, q_n*. We then say that* T *is the* $(L_n, q_n)_n$–tower.

iv) If $\mathcal{P}$ *is a predicate that a submodule of the first homology group of a surface may have, then we say that a homological tower of covers has* $\mathcal{P}$*, if each* L_n *has* $\mathcal{P}$.

In this article, we will usually assume without mention that all towers of covers are regular. In the setting of homological towers this is detected by the following.

Lemma 2.10. *Suppose that* T *is a* $(L_n, q_n)_n$*–tower. Then* T *is regular if and only if the image* $(L_n)_{q_n} < H_1(\Sigma_n; \mathbb{Z}/q_n\mathbb{Z})$ *of the submodule* L_n *is invariant under the action of the deck group* G_n.

Proof. We prove this inductively. Suppose we know that $\Sigma_n \to \Sigma$ is regular. The conjugation action of $\pi_1(\Sigma)$ on $\pi_1(\Sigma_n)$ is generated by the conjugation action of $\pi_1(\Sigma_n)$ on itself and the deck group action of G_n on $\pi_1(\Sigma)$ (with respect to any chosen basepoint).

Thus, the conjugation action of $\pi_1(\Sigma)$ on $H_1(\Sigma_n, \mathbb{Z})$ factors through the action of G_n on $H_1(\Sigma_n; \mathbb{Z})$. The next cover Σ_{n+1} is defined by a surjection $H_1(\Sigma_n; \mathbb{Z}) \to G$. The cover Σ_{n+1} to Σ will therefore by regular if and only if the kernel of this surjection is invariant under the action of G_n, which will be the case if and only if $(L^{(n)})_{q_n}$ is invariant under G_n. This shows the lemma. □

There are two complementary ways one can think of a (L_n, q_n)–tower. Suggested by the notation, the tower is completely determined by submodules L_n and the numbers q_n. That is, we think of the tower as an inductive sequence of choices of $L_n < H_1(\Sigma_n; \mathbb{Z})$ and q_n, which then already determines the next level Σ_{n+1} etc.

On the other hand, if we are just interested in the tower of covers, we can simply think of a sequence of finite covers $\Sigma_{n+1} \to \Sigma_n$, each of which is Abelian. In particular, if we fix the sequence q_n, then to determine the tower we actually only need to know the reductions $(L_n)_{q_n} < H_1(\Sigma_n; \mathbb{Z}/q_n\mathbb{Z})$ as opposed to the $\mathbb{Z}$–submodules.

Defintion 2.11. *Suppose that $T = (\Sigma_n)_n$ is a regular tower of covers. Let $\gamma \in \pi_1(\Sigma)$ be a loop. We denote by*

$$[\gamma]_n := [\hat{\gamma}^{k(n)}] \in H_1(\Sigma_n; \mathbb{Z})$$

the homology class defined by an elevation of γ. Here, $k(n)$ is the degree of lifting of γ for the cover Σ_n.

Note that $[\gamma]_n$ is only defined up to the action of the deck group G_n. This ambiguity will be irrelevant for the purposes of this article. In order to make lifts well-defined, one could choose consistent basepoints on each level of the tower, and replace "a lift" in the definition by "the lift at the preferred basepoint" (compare Section 6 for more on this).

Finally, we need to describe how the mapping class group $\mathrm{Mcg}(\Sigma)$ acts on towers of covers. We say that an element $\phi \in \mathrm{Mcg}(\Sigma)$ *preserves a tower T*, if ϕ lifts to a homeomorphism of each level Σ_n of T.

If the tower T is homological, then preserving it is a purely homological property in the following sense. Suppose that ϕ_* preserves $L_0 < H_1(\Sigma; \mathbb{Z})$. Then, by definition, ϕ lifts to a homeomorphism $\phi^{(1)}$ of Σ_1. Suppose now inductively that ϕ lifts to a homeomorphism $\phi^{(n)}$ of Σ_n. If $\phi_*^{(n)}$ preserves L_n, then it lifts to the next level Σ_{n+1}. We write this property as

$$\phi_*(L_*) = L_*$$

3. Detecting Meridians in Covers

In this section, we prove our first technical result: a way to detect meridians in a handlebody or compression body with a suitable homological tower of covers.

3.1. Handlebodies.

Let Σ be a closed surface of genus $g \geq 2$ and fix a homeomorphism

$$f : \partial V \to \Sigma$$

of Σ with the boundary ∂V of a genus g handlebody. We note that f is not unique; in this section we fix a specific choice of f once and for all. Also, fix any natural number $q > 1$. We restrict to the case of homological towers where all $q_n = q$ in this section; all results would remain valid also for the case of any sequence (q_n) with all $q_n > 1$. Towers with varying prime q_n can be useful; compare Section 5.2.

This identification of Σ with the boundary of a handlebody defines a homological tower of covers $\Sigma^{(n)}$. Namely, consider the family V_n of mod–q homology covers of the handlebody V. That is, the cover V_{n+1} of V_n is defined by the surjection

$$H_1(V_n; \mathbb{Z}) \to H_1(V_n; \mathbb{Z}/q\mathbb{Z}).$$

We let $\Sigma^{(n)} = \partial V_n$ be the boundary surface. Then $(\Sigma^{(n)})$ is a homological tower of covers, and the submodules $L^{(n)}$ defining this cover have a topological description:

$$L^{(n)} = \ker \left(H_1(\Sigma^{(n)}; \mathbb{Z}) \to H_1(V_n; \mathbb{Z}) \right).$$

Note that $L^{(n)}$ is a Lagrangian subspace of $H_1(\Sigma^{(n)}; \mathbb{Z})$ with respect to the algebraic intersection pairing, and that $L^{(n)}$ is generated by the homology classes of any cut system for V_n by Lemma 2.1.

For later use, we say that a Lagrangian tower of covers $(\Sigma^{(n)}, L^{(n)})_n$ is *geometric* if it arises from this construction.

The following theorem is a geometric proof of the fact that the free group $\pi_1(V)$ is residually finite (via towers of homology covers).

Theorem 3.1. *Let $\gamma \in \pi_1(\Sigma)$ be any loop. Suppose that for every n,*

$$[\gamma]_n \in L^{(n)} < H_1(\Sigma^{(n)}; \mathbb{Z}).$$

Then γ is nullhomotopic in V.

Proof. First, observe that under the assumption, actually γ lifts with degree 1 to every $\Sigma^{(n)}$. We prove this inductively. Suppose that γ lifts to a curve $\gamma^{(n)} \subset \Sigma^{(n)}$ with degree 1. Write

$$H_1(\Sigma^{(n)}; \mathbb{Z}) = L^{(n)} \oplus C^{(n)}$$

where the induced map $C^{(n)} \to H_1(V_n; \mathbb{Z})$ is an isomorphism. By assumption we have that $[\gamma^{(n)}] = [\gamma]_n \in L^{(n)}$, and thus γ lifts with degree 1 to $\Sigma^{(n+1)}$ as well.

To show the theorem, it suffices to show that there is some n, and a cut system $C_n \subset \Sigma^{(n)} = \partial V_n$ which is disjoint from a lift $\gamma^{(n)}$ of γ to ∂V_n. Namely, in that case, by Lemma 2.2, $\gamma^{(n)} \in \ker(\pi_1(\partial V_n) \to \pi_1(V_n))$ and thus $\gamma \in \ker(\pi_1(\partial V) \to \pi_1(V))$.

To this end, we will successively choose cut systems $C_n \subset \Sigma^{(n)}$ so that the intersection number of $\gamma^{(n)}$ with C_n strictly decreases in n. Recall that if $C_n \subset \Sigma^{(n)}$ is a cut system, then the cover $\Sigma^{(n+1)}$ is defined by requiring that algebraic intersection mod q with every element in C_n is zero.

Now, let $\hat{C}_{n+1}$ be the preimage of C_n in $\Sigma^{(n+1)}$. Let $k = \deg(\Sigma^{(n+1)} \to \Sigma^{(n)})$. Then the complement of $\hat{C}_{n+1}$ in $\Sigma^{(n+1)}$ consists of k components, each of which is a sphere with $2g$ boundary components; $g = g(\Sigma^{(n)})$ being the genus of $\Sigma^{(n)}$. Furthermore, every component of $\hat{C}_{n+1}$ is contained in (the closures of) two different complementary components.

Since $\gamma^{(n)}$ lifts to a curve $\gamma^{(n+1)} \subset \Sigma^{(n+1)}$, we have the following equality for geometric intersection numbers:

$$i(\gamma^{(n)}, C_n) = i(\gamma^{(n+1)}, \hat{C}_{n+1}).$$

Take $c \in \hat{C}_{n+1}$ any component which intersects $\gamma^{(n+1)}$. By the discussion above, $\hat{C}_{n+1} \setminus \{c\}$ is still a system of disks for the handlebody V_{n+1}, and

$$i(\hat{C}_{n+1} \setminus \{c\}, \gamma^{(n+1)}) < i(\hat{C}_{n+1}, \gamma^{(n+1)})$$

By Lemma 2.3, there is a cut system $C_{n+1} \subset \hat{C}_{n+1} \setminus \{c\}$ for V_{n+1}, which then satisfies the inductive hypothesis. The theorem follows by induction. □

The following corollary proves the first two assertions of Theorem 1 from the introduction.

Corollary 3.2. Let ψ be a mapping class of Σ. Then $\psi \in \mathcal{H}$ if and only if $\psi_* L^{(*)} = L^{(*)}$. Equivalently, $\psi \in \mathcal{H}$ if and only if ψ lifts to all $\Sigma^{(n)}$.

In fact, for any ψ there is a n which can be computed from ψ so that $\psi \in \mathcal{H}$ if and only if $\psi_* L^{(k)} = L^{(k)}$ for all $k \leq n$.

Proof. By Lemma 2.4, a mapping class ψ lies in the handlebody group if and only if for every meridian α, the image $\psi(\alpha)$ is also a meridian. If α is a meridian, then $[\alpha]_n \in L^{(n)}$ for all n. By the assumption on ψ, we then also have $[\psi(\alpha)]_n \in L^{(n)}$. By Theorem 3.1, this implies that $\psi(\alpha)$ is a meridian, and the first claim of the corollary follows. The equivalent formulation is immediate from the proof of Theorem 3.1, where only the fact that the curve in question lifts to all levels of the tower is used.

To show the second assertion, fix any cut system $\alpha_1, \dots, \alpha_g$ for V. Let $K = \sum_{s,t} i(\alpha_s, \psi(\alpha_t))$ be the total number of intersections between the cut system and its image under ψ. Following the induction in the proof of Theorem 3.1, we see that $\phi(\alpha_i)$ is certified to be a meridian in $\Sigma^{(n)}$ for $n \leq K$, if it lifts to $\Sigma^{(n)}$. By the second part of Lemma 2.4 the claim follows. □

Corollary 3.3. The handlebody group is seperable in the mapping class group.

Proof. Let $\Lambda^{(n)} < \mathrm{Mcg}(\Sigma)$ be the finite index subgroup consisting of all mapping classes which lift to $\Sigma^{(n)}$. By the previous corollary, we have

$$\bigcap \Lambda^{(n)} = \mathcal{H}.$$

□

As a corollary, we also obtain a slightly stronger version of a theorem of Koberda.

Theorem 3.4. *Consider the tower $(\{0\}, q)$ of q–homology covers for any q. There are submodules $L_1^{(n)}, L_2^{(n)} < H_1(\Sigma^{(n)};\mathbb{Z})$ invariant under the deck group G_n with the following property.*

If ψ is any infinite order element of $\mathrm{Mcg}(\Sigma)$, then for some n, the action of ψ on $H_1(\Sigma^{(n)})$ does not preserve both submodules $L_1^{(n)}, L_2^{(n)}$.

In particular, there is a cover in which ψ does not act like a deck group element.

Proof. Choose two identifications of Σ with the boundary of handlebodies V_1, V_2 so that no infinite order element of $\mathrm{Mcg}(\Sigma)$ extends to both V_1 and V_2. The existence of such identifications is well-known. A quick argument can be given as follows (compare also [Nam]): choose a pseudo-Anosov $\psi \notin \mathcal{H}$. Then the distance between the disk graph $\mathcal{D}(V_1)$ and its image $\psi^n\mathcal{D}(V_1)$ can be made arbitrarily big by increasing n. Acylindricity of the action of the mapping class group on the curve graph [Bow] now shows that no infinite order element of $\mathrm{Mcg}(\Sigma)$ can preserve both $\mathcal{D}(V_1)$ and $\psi^n\mathcal{D}(V_1)$.

Now, note that the homology covers $\Sigma^{(n)}$ cover the levels of the Lagrangian towers T_1, T_2 defined by both V_1 and V_2. Choose $L_1^{(n)}, L_2^{(n)}$ to be the preimages of the Lagrangians. The theorem now follows from Corollary 3.2: if lifts of ϕ to $\Sigma^{(n)}$ would always preserve both $L_1^{(n)}, L_2^{(n)}$, then ϕ would preserve the Lagrangian towers T_1, T_2, therefore extend to both V_1, V_2 and thus contradict the choice of handlebodies V_1, V_2. □

3.2. **Compression bodies.** Suppose that C is a compression body with outer boundary Σ (we choose some identification). There is a subspace

$$I^{(0)} = \ker\left(H_1(\Sigma;\mathbb{Z}) \to H_1(C;\mathbb{Z})\right)$$

generated by the boundaries of diskbounding curves. This subspace is still isotropic with respect to the algebraic intersection form, but will not be Lagrangian if C is not a handlebody (see Lemma 2.6). We can inductively define a sequence of covers defined by the kernels

$$\pi_1(C^{(n+1)}) = \ker\left(\pi_1(C^{(n)}) \to H_1(C^{(n)}; \mathbb{F}_{q_n}) \to H_1(C^{(n)}; \mathbb{F}_{q_n})/I^{(n)}\right)$$

Explicitly, choose a system of separating meridians δ_i as in Lemma 2.5, and a nonseparating set of meridians $\alpha_1, \ldots, \alpha_k$ for the handlebody component. The fundamental group $\pi_1(C^{(n+1)}) < \pi_1(C^{(n)})$ of the cover $C^{(n+1)}$ of $C^{(n)}$ consists of those loops on $C^{(n)}$ which have algebraic intersection number 0 mod q_n with all of the α_i, and every curve on the boundary of a interval bundle component. In particular the lifts of the δ_i cut $C^{(n+1)}$ into handlebodies and interval bundles, each of which is a mod–q_n homology cover of a corresponding component in $C^{(n)}$.

We prove the following proposition for simple loops for simplicity of exposition; it remains true for arbitrary loops as in Theorem 3.1.

Proposition 3.5. *Suppose that α is a simple closed curve so that $[\alpha]_n \in I^{(n)}$ for all n. Then α bounds a disk in C.*

Proof. We prove this result by induction as in the proof of Theorem 3.1, reducing intersection number with a suitable system of disks.

For the purposes of this proof, an *splitting system of meridians* on $C^{(n)}$ is a collection $\delta_1, \ldots, \delta_l$ of disjoint meridians so that: if $D_1, \ldots, D_l$ are disjoint disks bounded by the δ_i, then each component of $C^{(n)} - (D_1 \cup \cdots \cup D_l)$ is a handlebody or a trivial interval bundle over a closed oriented surface.

Pick a basepoint p and an orientation on α. Suppose we have a splitting system $S^{(n)}$ of meridians for $C^{(n)}$. Let $a^{(n)}$ be a subarc of a lift of α to $\Sigma^{(n)}$ with the following properties:

i) $a^{(n)}$ begins at a lift of p and is oriented to agree with the orientation of α.
ii) $a^{(n)}$ intersects the splitting system in at most two points.

As a first step, we will show that by increasing n we can in fact make $a^{(n)}$ larger (under the identification of a lift of α with α), until it is the whole of α.

So, assume that n is such that $a^{(n)}$ is a proper subarc of a lift of α. By maximality of $a^{(n)}$, there is a subarc $a_1 \subset a^{(n)}$ whose endpoints lie on elements of $S^{(n)}$. There are two cases, in which we will either change the system $S^{(n)}$, or pass to a further level in the tower. In both cases, the arc $a^{(n)}$ will not be maximal anymore.

Case 1: The endpoints of $a_1 \subset a^{(n)}$ lie on different curves $\delta, \delta' \in S^{(n)}$. In this case, we simply replace δ in $S^{(n)}$ by the band sum δ^* of δ and δ' along a_1. This is still a splitting system, and a_1 intersects it in one point. Hence a_1 is not maximal.

Case 2a: The endpoints of $a_1 \subset a^{(n)}$ lie on the same curve $\delta \in S^{(n)}$, but for some $m > n$ the endpoints of the lift $a_1 \subset a^{(m)}$ lie on different lifts of δ. In this case, we pass to level m of the tower (using the set of all elevations of $S^{(n)}$ as $S^{(m)}$), and argue as in Case 1.

Case 2b: The endpoints of $a_1 \subset a^{(n)}$ lie on the same curve $\delta \in S^{(n)}$, and for all $m > n$ the endpoints of the lift $a_1 \subset a^{(m)}$ lie on a common lift of δ.

In this case, we let $b, b' \subset \delta$ be the two embedded subarcs of δ which have the same endpoints as a_1. Then $\gamma = a_1 \cup b, \gamma' = a_2 \cup b'$ are simple closed curves which lift with degree 1 to each $\Sigma^{(m)}, m > n$. We claim that γ, γ' are meridians in the compression body $C^{(n)}$. If this is the case, then we can replace in $S^{(n)}$ the curve δ by γ and γ'. This will be a splitting system which has fewer intersections with $a^{(n)}$.

To show the claim, we have to distinguish which kind of component a_1 is contained in. If a_1 lies in a handlebody component, then γ, γ' are meridians by Theorem 3.1. Thus suppose that a_1 lies in the boundary of a bundle component $X \subset C$. By assumption X is homeomorphic to $[0, 1] \times \Sigma_h$ for some h. The fact that γ, γ' lift to all $\Sigma^{(m)}, m > n$ implies that their projections to Σ_h lift to every iterated homology cover of Σ_h. This is only possible if they are in fact nullhomotopic in Σ_h, and thus γ, γ' are meridians in X.

Hence, we may assume that n is big enough so that a lift of α is disjoint from a splitting system of meridians on $C^{(n)}$, and thus it is contained in the boundary of a handlebody or an interval bundle component. If it is contained in the boundary of a handlebody component, we can apply Theorem 3.1 to conclude that α is a meridian.

Otherwise, α is a curve which lifts with degree 1 to every iterated homology cover of a trivial interval bundle over a closed surface. Arguing as above, this is only possible if α is in fact trivial in that interval bundle. This shows the proposition. ☐

Exactly as Corollary 3.3 we also obtain the following.

Corollary 3.6. The subgroup of $\mathrm{Mcg}(\partial C)$ of all those diffeomorphisms which extend to the compression body C is separable.

4. The infinite nature of meridians

In light of Theorem 3.1, it is natural to ask if there is some fixed, finite cover ∂V_n whose homology can detect if a curve is a meridian or not. The following theorem answers this in the negative, even for simple closed curves.

Theorem 4.1. *For every $n > 0$ there is some simple closed curve α, so that α lifts to all $\partial V_k, k \leq n$, a lift of α to ∂V_n lies in L_n, but α is not a meridian.*

Theorem 4.1 is an immediate consequence of Theorem 3.1 and the following proposition, which finishes the proof of Theorem 1 from the introduction.

Proposition 4.2. *For each n, let Γ_n be the subgroup of $\mathrm{Mcg}(\Sigma)$ consisting of all ψ which lift to $\Sigma^{(n)}$ and whose lifts preserve $L^{(n)}$. Then*

$$\bigcap \Gamma_n = \mathcal{H}$$

but $\mathcal{H} < \Gamma_n$ has infinite index for all n.

Proof. The first claim is exactly Corollary 3.2.

Fix any n. Since $\Sigma^{(n)}$ is a finite cover, there is a finite index subgroup $\Lambda <$ $\mathrm{Mcg}(\Sigma)$ and a representation

$$\rho : \Lambda \to \mathrm{GL}(H_1(\Sigma^{(n)}; \mathbb{Z}))$$

so that for each $\phi \in \Lambda$, $\rho(\phi)$ is equal to the action of some lift of ϕ on $H_1(\Sigma^{(n)};\mathbb{Z})$.

Let $K = \ker(\rho)$. Suppose that the conclusion would be false. Then there would be a finite index subgroup $K' < K$ which is completely contained in the handlebody group $\mathrm{Mcg}(V)$.

We first claim that this is only possible if K is finite. Namely, consider the action of K on the sphere of projective measured laminations of Σ. The action of the mapping class group is minimal (e.g. as stable laminations of pseudo-Anosovs are dense), and thus the same is true for the action of the finite index subgroup Λ. If $K < \Lambda$ is normal and infinite, the action of K is minimal as well. As $K' < K$ is finite index, the same would be true for K'. However, the action of the handlebody group on the sphere of projective measured laminations admits a closed, invariant measure 0 subset by theorems of Masur and Kerckhoff [Mas2, Ker] (see also the appendix of [LM] for a correction of a gap in Kerckhoffs argument). This is a contradiction, and so K is finite.

In that case there would therefore be a finite index subgroup $\Gamma' < \Gamma_n$, so that the action of Γ' on $H_1(\partial V_n)$ is faithful. This is impossible. To see this, let δ_1, δ_2 be two nonhomologous, nonseparating meridians, and let δ be a meridian which bounds a pair of pants with δ_1, δ_2. Let ϕ be any mapping class group element which fixes δ_1, δ_2, but not δ. Then, as all meridians lift to ∂V_n with degree 1, a lift of δ and a lift of $\psi(\delta)$ are homologous. Thus, the Dehn twists T_δ and $T_{\psi(\delta)}$ are different, yet act in the same way on $H_1(\partial V_n)$, contradicting the defining property of Γ'. □

Proof of Theorem 4.1. We prove this theorem by contradiction. Suppose that there would be some n, so that a simple closed curve α is a meridian if and only if it lifts to ∂V_n and defines a homology class in L_n. Then, by Theorem 3.1, we would have $\mathrm{Mcg}(V) = \Gamma_n$ for the group Γ_n of Proposition 4.2. This violates the conclusion of that proposition. □

5. Characterising Towers of Lagrangians

In order to relate the criterion given by Proposition 4.2 to the question of compressability of mapping classes, one needs to answer the following.

Question 5.1. Give a (useful) criterion on the Lagrangian subspaces $L^{(n)} < H_1(\Sigma^{(n)})$ appearing in a Lagrangian tower of covers

$$\cdots \to \Sigma_n \to \cdots \to \Sigma_1 \to \Sigma$$

that determines if the tower is geometric.

Recall that *geometric* here means that the tower is defined by an identification of Σ with the boundary of a handlebody. There are a few obvious conditions a geometric tower needs to satisfy

i) For every n, the Lagrangian $L^{(n+1)}$ maps to $L^{(n)}$ under the covering map.
ii) For every n, the Lagrangian $L^{(n)}$ is invariant under the deck group G_n.
iii) For every n, the Lagrangian $L^{(n)}$ is a direct summand of $H_1(\Sigma^{(n)};\mathbb{Z})$.

It is also clear that these conditions do not suffice to guarantee that a tower is geometric. Namely, as the mapping class group is finitely generated, there are countably many identifications of Σ with the boundary of a handlebody, and therefore only countably many geometric towers of Lagrangians.

The remainder of this section describes various obstructions and criteria to answer Question 5.1.

5.1. **Generation Properties.** The first method to approach Question 5.1 relies on knowledge about loops representing homology classes in the tower.

Defintion 5.2. *Let $T = (\Sigma^{(n)}, L^{(n)})$ be a homological tower of covers. We define*

$$\Pi(T) = \{\gamma \in \pi_1(\Sigma) \,|\, [\gamma]_n \in L^{(n)} \ \forall n\}$$

We say that the tower T is nonempty *if $\Pi(T)$ is not the trivial group. We say T is* saturated*, if $\Pi(T)$ is nonempty and the image in $H_1(\Sigma;\mathbb{Z})$ generates $L^{(0)}$.*

Observe that Theorem 3.1 can be rephrased as saying: if T is geometric, then $\Pi(T)$ is saturated and in fact consists of exactly those loops which are trivial in the handlebody.

The rest of this section is concerned with deriving properties of $\Pi(T)$ which hold for arbitrary Lagrangian towers.

Lemma 5.3. *For any homological tower T, the group $\Pi(T)$ is normal in $\pi_1(\Sigma)$.*

Proof. Observe that the conjugation action of $\pi_1(\Sigma)$ on itself lifts in each $\Sigma^{(n)}$ to the deck group action. Thus, the claim of the lemma follows from the fact that the subspaces $L^{(n)}$ are invariant under the deck group of $\Sigma^{(n)} \to \Sigma$. □

Proposition 5.4. *If T is Lagrangian then $\Pi(T)$ is normally generated by powers of disjoint simple closed curves.*

Proof. By Maskit's planarity theorem [Mas1, Theorem 3], to show the conclusion about $\Pi(T)$, it suffices to show that the cover X corresponding to $\Pi(T)$ is planar. Equivalently, one needs to show that the algebraic intersection pairing on X vanishes identically (see e.g. [Hem]).

So, suppose that this is not the case. Then there are curves $\alpha, \beta \subset X$ which have nontrivial algebraic intersection number. By curve surgery, we may assume that there is no subarc $a \subset \alpha$ which intersects β only in its endpoints, and returns to the same side of β. We aim to show that for a large enough n, the images α_n, β_n in $\Sigma^{(n)}$ also have nontrivial algebraic intersection, thereby violating that they are contained in the Lagrangian $L^{(n)}$. We may choose a hyperbolic metric on Σ, and equip each $\Sigma^{(n)}$ and X with the lifted metric. Then, if we let α, β be geodesic representatives, they retain the property on subarcs, and their images $\alpha_i, \beta_i \subset \Sigma^{(i)}$ intersect minimally in their isotopy classes for each i.

As α_0, β_0 define elements in $L^{(0)}$, they have algebraic intersection number 0. Thus, there is some (immersed) subarc $a_0 \subset \alpha_0$ which intersects β_0 only at the endpoints, and returns to the same side of β_0. Let $b_0 \subset \beta_0$ be a subarc with the same endpoints as a_0.

Observe that the loop $a_0 \cup b_0$ (and its potential closed lifts) cannot be contained in all $L^{(n)}$. Namely, otherwise it would define an element of $\Pi(T)$, lift to X as a closed curve, and in that scenario α would have a returning arc with respect to β (as a subarc of the lift of $a_0 \cup b_0$), contradicting the choice of α, β.

Hence, there is some n, and lifts of α_n, β_n to $\Sigma^{(n)}$, so that the geometric intersection number between these lifts is strictly smaller than that of α_0, β_0. Also note that the geometric intersection number is not 0, as they lift to the intersecting curves α, β on X.

Since α_n, β_n define elements of $L^{(n)}$, we can inductively repeat the argument, and further decrease the intersection number. However, as there is no strictly decreasing, infinite sequence of natural numbers, this is a contradiction. □

Theorem 5.5. *A Lagrangian tower is determined by some handlebody V if and only if it is primitive and saturated.*

Proof. One direction is obvious. For the other, note that by the previous theorem, if the tower is nonempty, we have

$$\Pi(T) = \langle\langle a_1^{n_1}, \ldots, a_k^{n_k} \rangle\rangle$$

where the a_i are disjoint and simple.

Next, note that the complement of the a_i consists of bordered spheres, since the homology classes of the a_i need to span the Lagrangian submodule $L^{(0)}$ of $H_1(\Sigma; \mathbb{Z})$.

Thus, there is a handlebody V, in which all of the a_i bound disks. Let $T_V = (\Sigma_V^{(n)}, L_V^{(n)})$ be the tower of Lagrangians defined by that handlebody. Since the submodule $L^{(0)}$ is the same for T as for T_V, the first levels $\Sigma^{(1)}, \Sigma_V^{(1)}$ agree as well. As the a_i bound disks in V, we further have that $L^{(1)} \subset L_V^{(1)}$. Since both $L^{(1)}$ and $L_V^{(1)}$ are half-dimensional direct summands (by primitivity) of $H_1(\Sigma^{(1)}; \mathbb{Z})$, they are in fact equal.

Hence, by induction, we have $L_V^{(n)} = L^{(n)}$ for all n. □

Theorem 5.5 gives a complete answer Question 5.1 in terms of saturation. To give a purely homological criterion for saturation, one would need to understand the group $\Pi(T)$ more explicitly from the homological data. To this end, we propose the following questions, which are also interesting in their own right as questions about finite covers of surfaces. The first question is relevant to determine when $\Pi(T)$ is not the trivial group.

Question 5.6. Suppose that $T = (\Sigma_n, L_n)_n$ is a homological tower of covers. Characterise which sequences

$$x_n \in H_1(\Sigma_n; \mathbb{Z})$$

appear as $[\gamma]_n$ for some loop $\gamma \in \pi_1(\Sigma)$.

By Proposition 5.1, it would be enough to answer Question 5.6 for γ which are freely homotopic to simple closed curves; in other words:

Question 5.7. Suppose that $\Sigma' \to \Sigma$ is a regular cover. Characterise which classes $x \in H_1(\Sigma'; \mathbb{Z})$ are defined by elevations of simple closed curves in Σ.

This latter question has received some attention recently [FH2, FH1, KS, Bog].

5.2. **The action of the mapping class group.** In this section, we begin study a geometric tower of Lagrangians $L^{(n)}$ as representations of the deck group G_n. This will lead to a different answer to Question 5.1 in terms of the (ill-understood) action of the mapping class group on the homology of finite covers of surfaces.

Recall that by the Chevalley-Weil theorem we have that

$$H_1(\Sigma^{(n)};\mathbb{Z}) \cong \mathbb{Z}^2 \oplus \mathbb{Z}[G_n]^{2g-2}.$$

if g is the genus of the surface Σ. Similarly, for the homology of the cover V_n of the handlebody, whose boundary is $\Sigma^{(n)}$, we have

$$H_1(V_n,\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}[G_n]^{g-1}.$$

This follows from the Chevalley-Weil theorem for graphs (sometimes called Gaschütz theorem).

Recall that by normality of the tower of covers, the Lagrangians $L^{(n)}$ are invariant under the deck groups G_n. We thus have a commutative diagram of representations

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^{(n)} & \longrightarrow & H_1(\Sigma^{(n)};\mathbb{Z}) & \xrightarrow{i_*} & H_1(V_n;\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{\cong} & & \downarrow{\cong} & & \\ 0 & \longrightarrow & L^{(n)} & \longrightarrow & \mathbb{Z}^2 \oplus \mathbb{Z}[G]^{2g-2} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z}[G]^{g-1} & \longrightarrow & 0 \end{array}$$

(compare also [GLLM] for a discussion of this). As a consequence, by semisimplicity of representations of finite groups, we conclude that the Lagrangian $L^{(n)}$ as a representation of G_n is itself isomorphic to

$$L^{(n)} \cong \mathbb{Z} \oplus \mathbb{Z}[G_n]^{g-1}.$$

The fact that the tower $L^{(n)}$ is geometric means that there is an identification of Σ with the boundary of a handlebody

$$f_0 : \partial V_0 \to \Sigma$$

which determines

$$T = (\Sigma^{(n)}, L^{(n)})$$

be the corresponding tower of Lagrangian covers. Any other identification of Σ with the boundary of a handlebody differs from f by an element ϕ of the mapping class group of Σ. Hence, any other geometric tower of Lagrangian covers can be obtained from T by the action of the mapping class group.

Here, we think about a tower of Lagrangian subspaces as an inductive choice of Lagrangians $L^{(n)}$ which then determine the "next level" in the tower; a suitable subgroup of the mapping class group acts on the Lagrangians of each level.

The rest of this section is concerned with analysing which Lagrangians $L^{(0)}, L^{(1)}$ appear in geometric towers. We will see that the constraints are very mild.

Lemma 5.8. *Any Lagrangian summand L of $H_1(\Sigma;\mathbb{Z})$ appears as $L^{(0)}$ in a geometric tower of Lagrangians.*

Proof. It is well-known that the standard homology representation

$$\mathrm{Map}(\Sigma) \to \mathrm{Sp}(2g,\mathbb{Z})$$

is surjective. Furthermore, the integral symplectic group acts transitively on Lagrangian summands in $H_1(\Sigma;\mathbb{Z})$. This shows the lemma. □

If we fix $L^{(0)}$ and q_0, then the first cover $\Sigma^{(1)}$ of the Lagrangian tower is determined. The possible choices for $L^{(1)}$ can now be described in terms of the mapping class group action.

Lemma 5.9. *Let $T_V = (\Sigma_V^{(n)}, L_V^{(n)})$ be a tower of Lagrangian covers determined by a handlebody V. If $T = (\Sigma^{(n)}, L^{(n)})$ is any geometric tower of Lagrangian covers with $L^{(0)} = L_V^{(0)}$, then there is an element $\phi \in \mathcal{I}(\Sigma)$ so that*

$$L^{(1)} = \hat{\phi}_* L_V^{(1)}$$

where $\hat{\phi}$ is a lift of ϕ to $\Sigma^{(1)} = \Sigma_V^{(1)}$.

Proof. Let f' be any identification

$$f' : \partial V \to \Sigma$$

The composition $f' \circ f^{-1} : \Sigma \to \Sigma$ is a mapping class which preserves $L^{(0)}$ by assumption. Thus, $f' \circ f^{-1}$ can be written as a product $\Psi\Phi$ where $\Psi \in \mathcal{I}, \Phi \in \mathrm{Mcg}(V)$ [Hir]. Both Φ, Ψ lift to $\Sigma^{(1)}$. The lift of Φ perserves $L_V^{(1)}$, and the lemma follows. □

The action of lifts of elements in the mapping class group on the homology of Abelian covers have been studied by Looijenga in [Loo], and shown to be as large as possible in some sense. Namely, we have

Theorem 5.10 (Looijenga [Loo, (2.5)])**.** *Suppose that Σ is a closed surface of genus g, and G a finite Abelian group. If $g = 2$, also assume that the order of G is not divisible by 2 or 3. Let $\Sigma' \to \Sigma$ be a regular cover with deck group G. Denote by $\mathrm{Sp}_G(H_1(\Sigma'; \mathbb{Z}))$ the group of automorphisms which preserve the algebraic intersection pairing on $H_1(\Sigma'; \mathbb{Z})$ and commute with the G–action.*

Then lifts of elements in $\mathrm{Mcg}(\Sigma)$ generate a finite-index subgroup of $\mathrm{Sp}_G(H_1(\Sigma'; \mathbb{Z}))$.

Arguing as in Proposition 4.3 of [BBG$^+$], the conclusion actually remains true if one is only interested in lifts of elements in the Torelli group. This shows that "most" choices of Lagrangians $L^{(1)}$ which are as representations isomorphic to $\mathbb{Z}[G]^{g-1} \oplus \mathbb{Z}$ actually appear geometrically. See Corollary 5.12 below for an explicit version of this phenomenon in genus 2 (which is not covered by Looijenga's result stated above).

Continuing this line of analysis, one could now define a "cover Torelli group" as the subgroup of $\mathcal{I}$ formed by those elements which act trivially on $H_1(\Sigma^{(1)})$. Then, if one understood the action of this group on $H_1(\Sigma^{(2)})$, one could continue the analysis inductively. At this time, to the knowledge of the author, almost nothing is known about such cover Torelli groups. Theorems 3.4 and 4.1 imply that they form an infinite descending sequence of normal subgroups in the mapping class group.

In the remainder of this section, we will study in detail a concrete genus 2 example which sheds some light on the structure of $L^{(1)}$.

Take Σ a surface of genus 2, and let Γ be an embedded graph on Σ, with one vertex and edges a, b, c, d, so that $\Gamma \to \Sigma$ induces a surjection on $\pi_1(\Sigma)$ as in Figure 1. We denote the elements of the fundamental group defined by the edges by the same letters. The corresponding homology classes $[a], [b], [c], [d]$ are then a symplectic basis of $H_1(\Sigma; \mathbb{Z})$. We assume that Σ is the boundary of a handlebody V in which b, d are meridians.

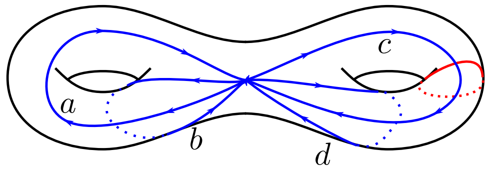

FIGURE 1. A basis for $\pi_1(\Sigma)$

We let $\Sigma^{(1)}$ be the cover induced by the mod–2 homology cover of V. Then, $\Sigma^{(1)}$ is a regular cover whose deck group is the Klein four group $G = \mathbb{Z}/2 \times \mathbb{Z}/2$. The map $\pi_1(\Sigma) \to G$ maps a to $(1, 0)$ and c to $(0, 1)$.

In this case, the Chevalley-Weil theorem takes a particularly simple form. Namely, we have

$$H_1(\Sigma^{(1)}; \mathbb{Q}) = \mathbb{Q}^4 \oplus W_1 \oplus W_{12} \oplus W_2$$

where W_1, W_2, W_{12} are each isomorphic to $\mathbb{Q}^2$ as vector spaces. G acts as $g \cdot v = \lambda_k(g)v$ where

$$\lambda_1(1, 0) = -1, \quad \lambda_1(0, 1) = 1$$
$$\lambda_{12}(1, 0) = -1, \quad \lambda_{12}(0, 1) = -1$$
$$\lambda_2(1, 0) = 1, \quad \lambda_2(0, 1) = -1$$

To study the action of lifts on the W_i, it is advantageous to consider certain subcovers. To do this, let $\Sigma_1 = \Sigma^{(1)}/\langle(0, 1)\rangle$, or, in other words, the cover obtained by algebraic intersection mod 2 with d. Then the transfer map

$$H_1(\Sigma_1; \mathbb{Q}) \to H_1(\Sigma^{(1)}; \mathbb{Q})$$

has image exactly $\mathbb{Q}^2 \oplus W_1$. In fact, by Chevalley-Weil we have

$$H_1(\Sigma_1; \mathbb{Q}) = \mathbb{Q}^4 \oplus W$$

and transfer induces an isomorphism of representations between W and W_1. Hence, to describe the action of the Torelli group on W_1, it suffices to compute the action of the Torelli group on W. Compare the discussion in and preceding Proposition 6.2 of [GL] as well as [Loo] for more on these reductions.

One can define covers Σ_2, Σ_{12} with analogous properties for the representations W_2, W_{12}.

The homology of the degree 2 cover Σ_1 can be decribed explicitly as

$$H_1(\Sigma'; \mathbb{Z}) = \mathbb{Z}^4 \oplus W_{\mathbb{Z}}$$

where $W_{\mathbb{Z}}$ is generated by $A = [a]_1 - (1, 0)[a]_1$ and $B = [b]_1 - (1, 0)[b]_1$. Here, as before, we denote by $[\gamma]_1$ the homology class in $H_1(\Sigma_1; \mathbb{Z})$ determined by a lift of the loop γ[1]

Proposition 5.11. *For any choice of $k \in \{1, 12, 2\}$ there are curves δ_1, δ_2 with the following properties:*

(1) δ_1, δ_2 are separating simple closed curves on Σ.
(2) Lifts of δ_1, δ_2 to Σ_l for $l \neq k$ are separating simple closed curves.

[1]there is an implicit choice of basepoint here, to make these things well-defined. The choice does not matter for the sequel.

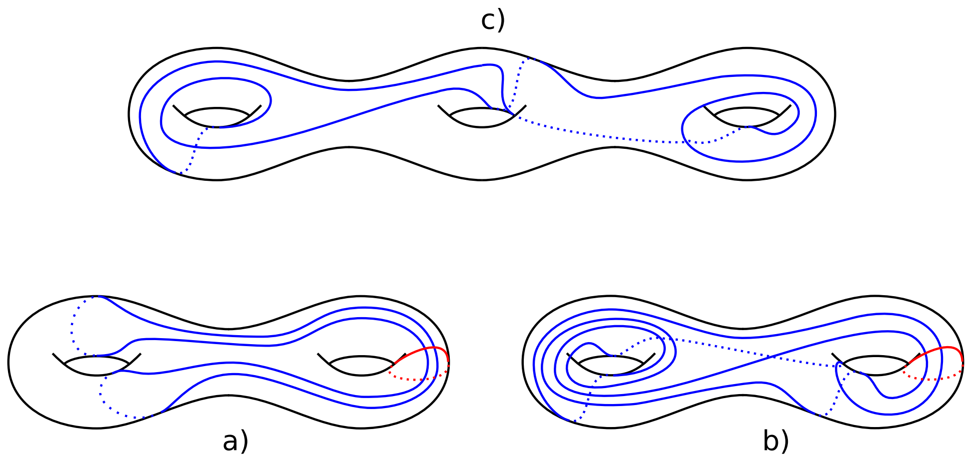

FIGURE 2. a) The curve δ_1, b) The curve δ_2, c) A lift of δ_2 to Σ_3

(3) The Dehn twists $T_{\delta_1}, T_{\delta_2}$ lift to mapping classes of Σ_1 which act on the subspace in $H_1(\Sigma_k, \mathbb{F}_3)$ spanned by the images of A, B as the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Proof. We give the construction for $k = 2$; the other cases are similar. In this case, we can explicitly construct the curves.

The desired curves are, written in the generating set given by Γ

$$\delta_1 = acbc^{-1}a^{-1}b^{-1}, \qquad \delta_2 = cda^2bd^{-1}c^{-1}a^{-1}b^{-1}a^{-1}$$

Both of these indeed lift to separating curves on Σ_1 and Σ_{12}. This can be seen by describing a lift in terms of the preimage of Γ (alternatively, one can compute Fox derivatives).

As an example, δ_1 lifts in Σ_1 to the sum of the lifts of $acbc^{-1}a^{-1}$ and b^{-1}. But, in Σ_1 both lifts of b are homologous. Thus, δ_1 indeed lifts to a nullhomotopic curve. In Σ_{12}, both of ac, b lift with degree 1, showing the result.

In Σ_2, lifts of δ_1, δ_2 define the homology classes

$$[\delta_1]_2 = [b]_2 - (0,1)[b]_2, \qquad [\delta_2]_2 = 2[a]_2 + [b]_2 - (0,1)(2[a]_2 + [b]_2)$$

Using the standard formula for the action of a Dehn twist ($[T_\alpha(\beta)] = \beta + i(\beta, \alpha)[\alpha]$), one can compute that these act on $W_2 \otimes \mathbb{F}_3$ as the matrices

$$M_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}.$$

From these the desired matrices can be obtained ($(M_2M_1)^2, M_1^2$). ☐

Corollary 5.12. In $H_1(\Sigma^{(1)}, \mathbb{F}_3)$, every subspace of the form

$$L^{(0)} \oplus \langle l_1 \rangle \oplus \langle l_{12} \rangle \oplus \langle l_2 \rangle \subset H_1(\Sigma^{(0)}, \mathbb{F}_3) \oplus W_1 \oplus W_{12} \oplus W_2$$

with all $l_k \neq 0$ appears as the image of $L^{(1)}$ for some geometric tower of Lagrangians.

Proof. Using Proposition 5.11, lifts of the Torelli group contain the group

$$\{\mathrm{Id}\} \times SL(2,\mathbb{F}_3) \times SL(2,\mathbb{F}_3) \times SL(2,\mathbb{F}_3)$$

acting on $H_1(\Sigma^{(0)},\mathbb{F}_3) \oplus W_1 \oplus W_{12} \oplus W_2$. Since $SL(2,\mathbb{F}_3)$ acts transitively on nonzero vectors in $\mathbb{F}_3^2$, the corollary follows. □

Corollary 5.13. Let Σ be a surface of genus 2. Suppose $q_1 = 2, q_2 = 3$ and $q_n > 1$ for $n > 2$. For any choice of

i) Lagrangian summand $L^{(0)}$ of $H_1(\Sigma;\mathbb{Z})$, and
ii) Lagrangian summand $L^{(1)}$ of $H_1(\Sigma^{(1)};\mathbb{F}_3)$ which, as a G_1–representation is isomorphic to $\mathbb{F}_3[G] \oplus \mathbb{F}_3$,

there is a geometric tower of Lagrangians beginning with $L^{(0)}, L^{(1)}$.

Proof. The first claim follows from Lemma 5.8. The second claim follows from Corollary 5.12, since $L^{(1)}$ is of the form in ii) if and only if it satisfies the assumption of that corollary. □

6. Profinite Perspective

In this section we interpret some of the results of previous sections in the framework of prohomology as studied e.g. by Koberda [Kob] and Boggi [Bog]. Our main contribution here is twofold: on the one hand, we are able to restrict to very specific covers (iterated homology covers, see below), and on the other hand our constructions are elementary and explicit.

The core idea is that a collection of covers of a surface Σ is a directed system, and thus the same is true for the homology of these covers. *Prohomology* is then simply the inverse limit of this system. If the system of covers is sufficiently large, then prohomology will capture geometric information about loops on the surface. Our objects and results are fairly close to [Bog], but the proofs have a more combinatorial and explicit flavour.

Defintion 6.1. *A* directed system of covers *is a set $\mathcal{Q} = \{f_n : \Sigma_n \to \Sigma\}$ of regular, finite covers of Σ, indexed by some set Q, so that the following holds.*

For any $n, m \in Q$ there is a $k \in Q$ so that Σ_k covers both Σ_n, Σ_m.

If $\mathcal{Q} = \{f_n : (\Sigma_n, p_n) \to (\Sigma, p)\}$ is a directed system of covers, then the homology groups $H_1(\Sigma_n; F)$ for any field F form a directed system of F–vector spaces in the usual sense, by taking the bonding maps to be the induced maps in homology.

Defintion 6.2. *If $\mathcal{Q}$ is a directed system of covers of Σ, then we define* prohomology *as*

$$\mathcal{H}_1^{\mathcal{Q}}(\Sigma; F) = \varprojlim H_1(\Sigma_n; F).$$

In hands-on terms, an element in $\mathcal{H}_1^{\mathcal{Q}}(\Sigma; F)$ is a compatible choice of homology classes in each $H_1(\Sigma_n; F)$. Note that prohomology is a F–vector space.

Useful families $\mathcal{Q}$ of covers include solvable, nilpotent, or p–covers for primes p [Kob, Bog]. We will use *iterated homology covers.* Namely, for any sequence of numbers $(q_n)_n, q_n > 1$ consider the $(\{0\}, q_n)$–homology tower (Σ_n). That is, Σ_{n+1} is the cover of Σ_n defined by the kernel of the map

$$\pi_1(\Sigma_n) \to H_1(\Sigma_n; \mathbb{Z}/q_n\mathbb{Z}).$$

Finally, we let $\mathcal{Q}$ be the directed system of covers formed by the (Σ_n) for any choice of sequences (q_n).

Next, we will define a way to encode loops $\gamma \in \pi_1(\Sigma)$ in $\mathcal{H}_1^{\mathcal{Q}}(\Sigma; F)$. The main issue in this endevour is that any given loop γ may not lift to some of the $H_1(\Sigma_n; F)$ and therefore does not seem to define a class in prohomology.

Koberda solves this issue in [Kob] by encoding the images of γ in the deck groups. We will use a different perspective (compare also [Bog]), motivated by observation that for any loop γ and any based finite cover $f_n : (\Sigma_n, p_n) \to (\Sigma, p)$ the cyclic group generated by γ defines a subspace in $H_1(\Sigma_n; F)$ (generated by the preferred elevation).

To this end, consider the tower of homology covers $\mathcal{Q}$, and choose in each cover from $\mathcal{Q}$ a basepoint, so that they map to each other under the covering maps. Define pointed prohomology as

$$\mathcal{H}_1^{\mathcal{Q}}(\Sigma, p; F) = \varprojlim_{\Sigma' \in \mathcal{Q}} H_1(\Sigma'; F).$$

As a vector space, this is the same as the usual prohomology defined above; we emphasise the basepoint as it is important in the next construction.

To formalise the encoding mentioned above, we use projectivisations. First, note that there are natural projection maps between (extended) projectivisations whenever $m > n$:

$$\mathbb{P}H_1(\Sigma_m; F) \cup \{0\} \to \mathbb{P}H_1(\Sigma_n; F) \cup \{0\}$$

We then define projectivised prohomology as the limit of those projectivisations

$$\mathbb{P}\mathcal{H}_1^{\mathcal{Q}}(\Sigma, p; F) = \varprojlim_{\Sigma' \in \mathcal{Q}} (\mathbb{P}H_1(\Sigma'; F) \cup \{0\}) \setminus \{0\}.$$

There are projection maps

$$\mathbb{P}\mathcal{H}_1^{\mathcal{Q}}(\Sigma, p; F) \to \mathbb{P}H_1(\Sigma'; F) \cup \{0\}$$

for all $\Sigma' \in \mathcal{Q}$.

Defintion 6.3. *Let $\mathcal{Q}$ be a directed system of covers of (Σ, p). Then there is a map*

$$\iota : \pi_1(\Sigma, p) \to \mathbb{P}\mathcal{H}_1^{\mathcal{Q}}(\Sigma, p; F).$$

For any $\Sigma' \in \mathcal{Q}$, the elevation $[\gamma]_{\Sigma'}$ is an element of the projection of $\iota(\gamma)$ in $H_1(\Sigma')$.

The mapping class group $\mathrm{Mcg}(\Sigma, p)$ of homeomorphisms which fix the point p acts on $\mathbb{P}\mathcal{H}_1^{\mathcal{Q}}(\Sigma, p; F)$ if $\mathcal{Q}$ consists of characteristic covers. In that case, the map ι is equivariant.

With this in hand, we can rephrase Theorem 3.1 as follows.

Theorem 6.4. *Suppose that Σ is identified with the boundary of a handlebody V. Then there is a subset*

$$\mathcal{L} \subset \mathbb{P}\mathcal{H}_1^{\mathcal{Q}}(\Sigma, p; F)$$

so that a loop $\gamma \in \pi_1(\Sigma)$ is trivial in $\pi_1(V)$ if and only if

$$\iota(\gamma) \in \mathcal{L}.$$

Similarly, one could characterise the handlebody group (which fixes p) as the subgroup of the mapping class group (which fixes p) which preserves $\mathcal{L}$.

We finish by reproving some results which appear in [Bog] using combinatorial methods, and in a more explicit way. We begin by the following observation, which is a homology-cover version of a famous theorem of Scott [Sco1, Sco2]. A version for p–group covers is proved as [Bog, Theorem 3.3] with different methods. Compare also [MP] for methods to construct covers more similar to our treatment here.

Lemma 6.5. *Let $x \in \pi_1(\Sigma, p)$ be any loop. Then there is some iterated homology cover Σ' so that the elevation of x to Σ' does not have transverse self-intersections.*

Proof. First note that it suffices to show that there is a finite tower $\Sigma^{(n)} \to \Sigma^{(n-1)} \to \cdots \to \Sigma$ of successive Abelian covers so that x has a simple lift to $\Sigma^{(n)}$. This is because there will be some iterated homology cover Σ' which covers $\Sigma^{(n)}$, and x will have simple elevations to that Σ' (being elevations of a simple curve on $\Sigma^{(n)}$).

We will show the existence of the desired tower of covers inductively, successively decreasing the self-intersection number of x. In order to do so, we use the following elementary observations.

Observation 1. Let x be a loop, and suppose $y \subset x$ is a simple subloop. If $\Sigma' \to \Sigma$ is a cover so that x lifts to Σ' but y does not, then a lift x' of x has smaller self-intersection number than x.

Observation 2. Let y be a separating simple closed curve on Σ. Then there is a finite Abelian cover of Σ so that elevations of y to Σ' are nonseparating.

Observation 3. Suppose that y, z are two disjoint curves, so that either z is nullhomologous, or homologous to y. Then there is an order 2 cover to which y, z lift, but lifts are not homologous, and not nullhomologous.

The induction has two cases

Case 1. $[x] = 0$. Suppose that there is any simple subloop $y \subset x$ with $[y] \neq 0$. Then there is a cyclic cover to which y does not lift. As $[x] = 0$, it does lift, and so by Observation 1, self-intersection number decreases.

If there is no simple subloop with $[y] \neq 0$, take a cover as in Observation 2 which makes a separating subloop y nonseparating. In that cover, we either are not in Case 1 anymore, or the previous argument applies.

Case 2. $[x] \neq 0$. It is well known (see e.g. [FM, Proposition 6.2]) that every homology class in $H_1(\Sigma; \mathbb{Z})$ is the multiple of some class which is realised by a simple closed curve. Thus, there exists a collection of curves α_i, β_i intersecting in the standard pattern, so that x defines a multiple of $[\alpha_1]$. In particular, it has algebraic intersection nonzero only with β_1.

Take $y \subset x$ a simple subloop with $[y] \neq 0$. If y has nonzero algebraic intersection with any α_i or $\beta_j, j \neq 1$, then there is an Abelian cover to which x lifts and y does not, and we are done. Otherwise $[y] = \pm[\alpha_1]$, and we can assume that y is freely homotopic to α_1.

If $[x]$ is a proper multiple of $[y]$, then there is a cover to which x lifts but y does not, and we are done.

Otherwise, consider a minimal subloop $z \subset x$ which intersects y only in its endpoints, and $z' \subset z$ a simple subloop. z' is disjoint from α_1. If $[z'] \neq 0$ and $[z'] \neq \pm[y]$, then there is a cover to which z' does not lift, but x does, and we are done. Otherwise, apply Observation 3; in the new cover this situation will then not occur again. □

The following is the analog of Lemma 5.2 in [Kob] and Theorem 5.1 [Bog].

Lemma 6.6. *For $\mathcal{Q}$ the family of iterated homology covers, the map*

$$\iota : \pi_1(\Sigma, p) \to \mathbb{P}\mathcal{H}_1^{\mathcal{Q}}(\Sigma, p; F)$$

has the property that

$$\iota(x) = \iota(y)$$

if and only if x, y are powers of a common element $\gamma \in \pi_1(\Sigma, p)$.

Proof. By Lemma 6.5, we may assume that for some Σ', the preferred elevations $x', y' \subset \Sigma'$ are simple. If in fact the elevations of x', y' are disjoint, then upon taking a further cover they become nonhomologous, unless they are equal.

If they are not disjoint, we can argue as in the proof of Lemma 6.5 to take further covers which decrease intersection number between elevations until they are disjoint; the role of simple subloops will be played by simple loops embedded in $x' \cup y'$. □

Question 5.6 which has arose in the study of saturated towers of Lagrangians becomes in this language the following:

Question 6.7. Characterize the image of ι.

Note that by Lemma 6.5, it would be enough to characterise the image of simple closed curves under ι; the general answer would then follow as the union of the simple loops over all $\Sigma' \in \mathcal{Q}$. This is another reason why Question 5.6 reduces to Question 5.7.

As in Proposition 5.1, cover towers are well-adapted to detect simplicity of curves. The following is the analog of Corollary 2.4 of [Bog].

Lemma 6.8. *γ is freely homotopic to the power of a simple closed curve if and only if $\iota(\gamma)$ generates an isotropic subspace at each level.*

Sketch of proof. One direction is clear. The other direction follows as in the proof of Lemma 6.5, reducing intersection number between two simple elevations to 1 if they are not disjoint. □

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