(UN)DISTORTED STABILISERS IN THE HANDLEBODY GROUP

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Abstract. We study geometric properties of stabilisers in the handlebody group. We find that stabilisers of meridians are undistorted, while stabilisers of primitive curves or annuli are exponentially distorted for large enough genus.

1. Introduction

The handlebody group $\mathrm{Mcg}(V)$ is the mapping class group of a 3–dimensional handlebody. In this article, we study the subgroup geometry of stabilisers in $\mathrm{Mcg}(V)$ of meridians and primitive curves. A curve δ on the boundary of a handlebody is called a *meridian*, if it is the boundary of an embedded disc. A curve β is called *primitive*, if there is a meridian δ which intersects β in a single point.

Recall that a finitely generated subgroup $H < G$ of a finitely generated group G is *undistorted* if the inclusion homomorphism is a quasi-isometric embedding. In contrast, we say that it is *exponentially distorted*, if the word norm in H can be bounded by an exponential function of word norm in G, and there is no such bound of sub-exponential growth type. We refer the reader to e.g. [Far] for details on distortion functions of subgroups.

Our main results are:

Theorem 1.1. *Suppose that V is a handlebody or compression body, and that δ is a (multi)meridian in V. Then the stabiliser of δ in the handlebody group $\mathrm{Mcg}(V)$ is undistorted.*

Theorem 1.2. *Let V be a handlebody of genus g.*

i) Suppose that α is a primitive curve and $g \geq 3$. Then the stabiliser of α is exponentially distorted in $\mathrm{Mcg}(V)$.

ii) Suppose $A \subset V$ is a properly embedded annulus so that ∂A consists of primitive curves. Assume that $g \geq 3$ (if A is non-separating) or $g \geq 4$ (if A is separating). Then the stabiliser of A is exponentially distorted in $\mathrm{Mcg}(V)$.

To put these theorems into context, observe that the handlebody group is directly related to mapping class groups of surfaces (via restriction of homeomorphisms to the boundary) and to the outer automorphism group of free

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groups (via the action on the fundamental group). However, neither of these connections is immediately useful to study the geometry of $\mathrm{Mcg}(V)$: The inclusion $\mathrm{Mcg}(V) \to \mathrm{Mcg}(\partial V)$ may distort distances exponentially [HH1], and the kernel of the map $\mathrm{Mcg}(V) \to \mathrm{Out}(\pi_1(V))$ has an infinitely generated kernel [Luf, McC1]. In other words, there is no a-priori reason to expect that $\mathrm{Mcg}(V)$ shares geometric features with surface mapping class groups or outer automorphism groups of free groups.

However, it seems that its geometry nevertheless resembles that of outer automorphism groups of free groups. A first instance of this was the computation of its Dehn function in [HH2]: these are exponential for handlebody groups of genus at least three, just like those of $\mathrm{Out}(F_n)$ for $n \geq 3$.

The two main results of this article provide further evidence for this philosophy. In the surface mapping class group $\mathrm{Mcg}(\partial V)$, stabilisers of curves are undistorted for all curves (this follows e.g. immediately from the distance formula of Masur-Minsky [MM]). On the other hand, Handel and Mosher [HM] found that in the outer automorphism groups of free groups there is a dichotomy – stabilisers of free splittings are undistorted, whereas stabilisers of primitive conjugacy classes (and most other free factors) are exponentially distorted. By the van-Kampen theorem, the stabiliser of a meridian maps exactly to the stabiliser of a free splitting, while the stabiliser of a primitive curve maps to the stabiliser of a primitive conjugacy class.

The stabiliser of an annulus in the handlebody group maps to the stabiliser of a cyclic splitting in $\mathrm{Out}(F_n)$. Here, an exponential lower bound for distortion follows from the results in [HM] (although it is not explicitly discussed in that reference). We extend this by a similar upper bound, which shows that it has the same behaviour as in the handlebody group case.

Proposition 1.3. *Let $n \geq 4$ be given. Then the stabiliser of a cyclic splitting in $\mathrm{Out}(F_n)$ is exponentially distorted in $\mathrm{Out}(F_n)$.*

To prove Theorem 1.1, we define a projection of disc systems of V to disc systems in sub-handlebodies. This is carried out in Section 3. We recall the well-known fact that usual Masur-Minsky subsurface projections of meridians to sub-handlebodies are usually not meridians again (compare e.g. [Hen, Section 10]), and so our projection procedure is more involved and depends on choices. The lower bounds in Theorem 1.2 are proved by a reduction to the theorem by Handel-Mosher on stabilisers in the outer automorphism group of free groups. Here, the key difficulty is to realise certain free group automorphisms considered by Handel-Mosher as homeomorphisms of handlebodies with comparable word norm. This uses an idea which was already employed in [HH2]. The upper distortion bounds in Theorem 1.2 follow from a surgery procedure.

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2. Preliminaries

In this section we collect some basic tools on surfaces, handlebodies, and compression bodies that we will use throughout.

2.1. **Surface Basics.** Suppose that S is a surface. All *curves* are assumed to be simple and essential. We usually identify a curve with the isotopy class it defines. A *subsurface* will always mean a (possibly disconnected) submanifold with boundary no component of which is an annulus. Unless specified explicitly, we will assume that all curves and subsurface are in minimal position with respect to each other; compare [FM]. If α is a curve and Y is a subsurface, then we call the components of $Y \cap \alpha$ the *α-arcs with respect to Y*.

If δ is a multicurve, then we denote by $S - \delta$ the complementary subsurface. We will often call the intersections $\alpha \cap (S - \delta)$ the *α-arcs with respect to δ*.

If α is a curve, then we denote by

$$\pi_{S-\delta}(\alpha)$$

the *subsurface projection*, i.e. a maximal subset of non-homotopic arcs in the set $(S - \delta) \cap \alpha$ of α-arcs of δ.

2.2. **Handlebodies and Surgeries.** A *handlebody* is the 3-manifold obtained by attaching three-dimensional one-handles to a 3–ball. A *compression body* is the 3–manifold obtained from a handlebody by taking connect sums with a finite number of trivial surface bundles. A compression body has an *outer boundary component*, which is the unique boundary component of highest genus.

Suppose that V is a handlebody or compression body. A *meridian* is a simple closed curve on ∂V which bounds a disc in V. A *filling disc system* is a collection $\Delta = \{\delta_1, \ldots, \delta_k\}$ of disjoint meridians, bounding disjoint discs D_i with the property that each component of $V - \cup D_i$ is either a 3–ball or a trivial surface-bundle.

Next, we describe *surgery.* Namely, we recall the following standard result (compare e.g. the proof of Theorem 5.3 in [McC2]).

Lemma 2.1. *Suppose that Δ is a filling disc system, and that α is any meridian. Then there is a subarc $a \subset \alpha$, called a* wave*, with the following properties:*

i) $a \cup \Delta$ consists of two points on the same curve $\delta \in \Delta$. Call the components of $\delta - a = \delta_- \cup \delta_+$. Then the set

$$\{a \cup \delta_-, a \cup \delta_+\} \cup \Delta \setminus \{\delta\}$$

is a filling disc system.

ii) If V is a handlebody, then there is $\delta_* = \delta_\pm$ so that

$$\{a \cup \delta_*\} \cup \Delta \setminus \{\delta\}$$

is a filling disc system.

In the case of handlebodies, any arc a which intersects Δ only in its endpoints and returns to the same side of a curve in Δ is a wave.

We call the result of i) *full surgery*, and the result of ii) *surgery.*

The *handlebody group* is the mapping class group of a handlebody V. Equivalently, the handlebody group is the image of the restriction map

$$\mathrm{Mcg}(V) \to \mathrm{Mcg}(\partial V),$$

which is the group of those surface mapping classes of ∂V which extend to V. Similarly, the *compression body group* is either the mapping class group of a compression body C, or the subgroup of the mapping class group of the outer boundary component formed by those classes that extend to C.

3. Meridian Stabilisers

In this section we analyse stabilisers of meridians in handlebody and compression body groups. For simplicity of exposition we focus on the proof in the case of handlebody groups, and only indicate the necessary modifications in the case of compression body groups at the end of the section.

From an algebraic perspective, the study of meridian stabilisers reduces to the study of point-pushing and handlebody groups of smaller genus, just as in the case of surface mapping class groups; compare [Hen] for details.

The main result of this section is the following theorem:

Theorem 3.1. *Let V be a handlebody, and δ be a multimeridian. Then the stabiliser of δ in the handlebody group is undistorted.*

3.1. Models.

We will use two geometric models for the handlebody group in the proof of Theorem 3.1 (and subsequent arguments).

Definition 3.2. For numbers $k_0, k_1 > 0$, define a graph $\mathcal{G}(k_0, k_1)$ with

- **Vertices:** corresponding to pairs (C, l) of a filling meridian system C and a simple diskbusting loop l (up to isotopy), so that $i(C, l) \leq k_0$.
- **C-Edges:** between vertices (C, l) and (C', l) if C' is disjoint from C.
- **l-Edges:** between vertices (C, l) and (C, l') if $i(l, l') \leq k_1$.

A standard argument (compare [HH1, Lemma 7.3]) shows that there are choices of k_0, k_1 so that $\mathcal{G}(k_0, k_1)$ is connected. Once this is the case, the handlebody group acts on $\mathcal{G}(k_0, k_1)$ properly discontinuously and cocompactly. Hence, by the Svarc-Milnor lemma, $\mathcal{G}(k_0, k_1)$ will then be equivariantly quasi-isometric to the handlebody group. Throughout, we make a choice of k_0, k_1 with these properties and simply denote the resulting graph by $\mathcal{G}$.

Definition 3.3. For a multimeridian δ, we let $\mathcal{G}(\delta)$ be the full subgraph spanned by all those vertices (C, l) whose meridian system contains δ.

If we choose k_0, k_1 large enough, the subgraphs $\mathcal{G}(\delta)$ will be connected for all δ, and therefore $\mathcal{G}(\delta)$ is equivariantly quasi-isometric to the stabiliser of δ in $\mathcal{H}$. Hence, in order to prove Theorem 3.1, it suffices to show that the subgraph $\mathcal{G}(\delta)$ is undistorted in $\mathcal{G}$.

In the proof it will be useful to use a second model[1], which is very similar to the graph of rigid racks employed in [HH1].

Definition 3.4. For numbers k_0, k_1, the graph $\mathcal{R}(k_0, k_1)$ has

- **Vertices:** corresponding to (isotopy classes of)connected graphs $\Gamma \subset \partial V$ with at most k_0 vertices, which contain a filling meridian system as an embedded subgraph, and have simply connected complementary regions.
- **Edges:** between graphs Γ, Γ' which intersect in at most k_1 points.

As above, we can choose the constants k_0, k_1 so that the resulting graph is connected, and we will do so. Similarly, we define $\mathcal{R}(\delta)$, and assume that it is also connected.

There is a natural map
$$U : \mathcal{G} \to \mathcal{R}$$
which sends a vertex (C, l) to the union $C \cup l$ (assuming that l is in minimal position with respect to C). By choosing the constants defining $\mathcal{R}$ large enough, we may assume that this map is a quasi-isometry, and the same is true for the restriction
$$U : \mathcal{G}(\delta) \to \mathcal{R}(\delta)$$
The strategy to prove Theorem 3.1 is to start with any path $\gamma : [0, n] \to \mathcal{G}$, and to "project" it to a path in $\mathcal{R}(\delta)$ joining $U(\gamma(0))$ to $U(\gamma(n))$, taking care that the length of the projected path is coarsely bounded by n. Since U is a quasi-isometry, this will imply that $\mathcal{G}(\delta)$ is undistorted in $\mathcal{G}$, showing Theorem 3.1.

3.2. **Patterns and Surgery.** In this section we will describe a systematic way to simplify a cut system until it is disjoint from a given (multi)meridian δ, which is assumed to be fixed throughout the section. This will be the core ingredient used to project paths in $\mathcal{G}$ to $\mathcal{R}(\delta)$. To do so, we use throughout the following terminology and setup. Cut the boundary surface ∂V of the handlebody at the meridian δ. The resulting (possibly disconnected) surface with boundary has boundary components δ^+, δ^- corresponding to the two sides of δ.

Suppose now that C is a cut system intersecting δ transversely and minimally. Then we call, by a slight abuse of notation, the set
$$(\delta^+ \cap C) \cup (\delta^- \cap C)$$

[1]The reason for using both models is that curves and curve pairs have easier to phrase minimal position properties as opposed to embedded graphs.

the *intersection points of C with δ*. Recall that we call the connected components of $C \cap (S - \delta)$ the *C–arcs*. Note that endpoints of C-arcs are intersection points of C with δ and vice versa.

An *interval* will mean a subarc I of $\delta^+ \cup \delta^-$ whose endpoints lie in C. Observe that for the two endpoints x, y there are uniquely determined C–arcs γ_x, γ_y which intersect I in x, y (note that these arcs may coincide). We call them *C–arcs adjacent to I*.

Definition 3.5. A *partial pattern* for C is a collection $\mathcal{I}$ of intervals satisfying the following properties:

- **N):** Any two $I, J \in \mathcal{I}$ are disjoint or nested.
- **P):** If γ is a C–arc adjacent to some I, then both endpoints of γ are endpoints of intervals in $\mathcal{I}$.

Definition 3.6. A *chain* of a partial pattern $\mathcal{I}$ is a minimal length sequence $c = (I_1, \gamma_1, I_2, \gamma_2, \ldots, \gamma_k, I_1)$ of intervals and C–arcs so that for each i, the right endpoint of I_i is one endpoint of γ_i, and the other endpoint of γ_i is the left endpoint of I_{i+1}.

An easy induction, using property P), shows that every interval in a pattern is part of a unique chain. By concatenating the I_i and γ_i in a chain c, we obtain a closed loop on the surface, which by abuse of notation we also call a chain of the pattern. This concatenation is not embedded, but we can homotope it into *push-off position*, by pushing the intervals I_i slightly off of $\delta^+ \cup \delta^-$ into $S - \delta$. Property N) guarantees that we can choose push-off positions for all chains so that the chains themselves are simple closed curves, and different chains do not intersect (push off more deeply nested intervals further off of $\delta^+ \cup \delta^-$). From now on we assume that, unless specified explicitly, all chains are in push-off position.

A chain may be an inessential curve. To avoid this, we put

$$C(\mathcal{I}) = \{\alpha | \alpha \text{ is an essential curve defined by a chain of } \mathcal{I}\}.$$

Each $c \in C(\mathcal{I})$ is a concatenation of subarcs of C and (pushed off copies of) intervals in $\mathcal{I}$. We call the intervals $I \in \mathcal{I}$ which appear in this way *active* (intervals may be inactive if they lie on inessential chains defined by the pattern).

Definition 3.7. A *pattern* is a partial pattern $\mathcal{I}$ which additionally satisfies:

- **F):** The set $C(\mathcal{I}) \cup \{\delta\}$ forms a filling disc system for V.

Definition 3.8 (Compatible Patterns)**.** If C' is disjoint from C, then a pattern $\mathcal{I}$ for C and a pattern $\mathcal{I}'$ for C' are *compatible*, if any intervals $I \in \mathcal{I}$ and $I' \in \mathcal{I}'$ are either disjoint or nested.

Arguing as before with push-off representatives, we immediately obtain the following:

Lemma 3.9. *If C, C' are disjoint cut systems, and $\mathcal{I}, \mathcal{I}'$ are compatible patterns, then $C(\mathcal{I})$ and $C'(\mathcal{I}')$ are disjoint.*

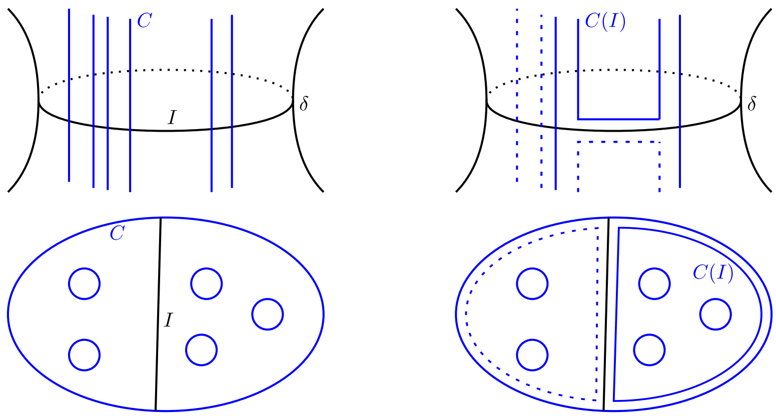

FIGURE 1. Push-off representatives of a surgery move.

Our first goal is to show that patterns always exist, and that for disjoint cut systems there are compatible patterns. This will be done by a standard surgery procedure, and the following lemma is analogous to various results in the literature, compare e.g. [HH1, Lemma 5.4], [Hem, Lemma 1.3] or [Mas, Lemma 1.1].

Lemma 3.10 ((Compatible) patterns exist)**.** *i) For any C in minimal position with respect to δ, there is a pattern $\mathcal{I}$ for C, which we call a* surgery pattern.

ii) If $\mathcal{I}$ is a surgery pattern[2] *for C, and C' is disjoint from C, then there is a surgery pattern for C' which is compatible with $\mathcal{I}$.*

Proof. i) We find the pattern inductively. Note that if an interval I is innermost, i.e. it does not contain any other intersection points of C with δ, then it defines an δ–arc with respect to C. Thus it makes sense to talk about intervals being a wave of δ with respect to C. Choose an interval I_1 which is a wave w of C with respect to δ and set $C = C_1$. Let C_2 be the result of the cut system surgery of C_1 at w. In fact, C_2 has a push-off representative as well, so that each curve in C_2 is a concatenation of parts of C_1 and the interval I_1 (compare Figure 1). If we choose I_1 outermost amongst all waves parallel to w, we can ensure that $C_1(I_1)$ is in minimal position with respect to δ. Inductively repeat the procedure, defining a surgery sequence C_i and a sequence of intervals I_j, so that each curve in each C_i is obtained as a concatenation of arcs in C and intervals $I_k, k \leq i$. The final term C_n of this surgery sequence is a cut system disjoint from δ. Let $\mathcal{I}$ be the set of those intervals I_k which are still part of C_n. By construction, we then have that $C(\mathcal{I}) = C_n$, and thus $\mathcal{I}$ is indeed a pattern.

ii) Suppose that C_i, I_i are the sequences constructed in part i) to form $\mathcal{I}$. We build I'_i and C'_i inductively. Suppose that $I'_j, C'_j, j \leq s$ are defined,

[2] i.e. it is obtained by applying part i) of this lemma

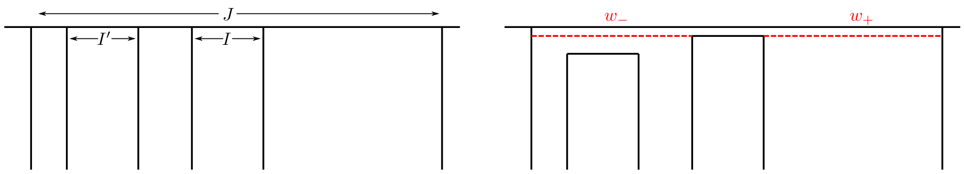

Figure 2. Nested intervals in a pattern define wings. If $I \subset J$ are directly nested, the wings can be made disjoint from the system $C(\mathcal{I})$, even if there are other intervals $I' \subset J$.

so that $I_i, i \leq r, I'_j, j \leq s$ are nested or disjoint, and so that C_r, C'_s are disjoint. Now consider I_r, and distinguish two cases. If I_r is disjoint from C'_s, then C_{r+1} is also disjoint from C'_s, and I'_r is nested or disjoint from the $I'_j, j \leq s$. Hence, $(r+1, s)$ satisfies the inductive hypothesis.

Alternatively, if I_r is not disjoint from C'_s, then there is a sub-interval $I'_r \subset I_r$ which defines a wave for C'_s. In that case, let C'_{s+1} be the surgery of C'_s at that sub-interval, and note that it is disjoint from C_r. Hence, $(r, s+1)$ satisfies the inductive hypothesis. By induction, the result follows.

□

The surgery patterns produced by Lemma 3.10 are not yet sufficient for our purposes. We will need a second move to improve patterns; to describe it we use the following terminology:

Suppose that $\mathcal{I}$ is a pattern for C, and suppose that $I \subsetneq J$ are two active intervals in $\mathcal{I}$. The two segments $w_- \coprod w_+ = J \setminus \operatorname{int} I$ can be interpreted (up to a small homotopy) as arcs with endpoints on $C(\mathcal{I})$, and we call these arcs *wings*; compare Figure 2. Observe that the two wings defined by a nested pair of intervals are homotopic as arcs with endpoints sliding on C. We say that I, J are *directly nested* if there is no active I' with $I \subsetneq I' \subsetneq J$. Suppose now that $I \subsetneq J$ are directly nested. In that case, wings define arcs that intersect $C(\mathcal{I})$ at most in their endpoints (compare again Figure 2 for this situation), since intersections of a wing with intervals $\hat{I}$ of the form $\hat{I} \subset J - I$ can be removed up to homotopy.

Suppose now that D is any filling disc system, and that w is an arc which has both endpoints on the same curve $\gamma \in D$. Then w defines an element in $\pi_1(V)$ by connecting the endpoints of w in any way along γ (observe that since γ is a meridian, it does not matter how we connect the endpoints). We say that w is a *V-trivial arc* if it has both endpoints on the same curve γ and additionally it defines the trivial element in $\pi_1(V)$.

Lemma 3.11. *If D is a filling disc system and w is a V–trivial arc, then it contains a subarc $w_0 \subset w$ which defines a wave with respect to D.*

Proof. Choose a subarc $d \subset D$ so that $d * w$ is a meridian. Lift $d * w$ to the boundary Y of the universal cover of V. Note that since this lift is a simple closed curve, a lift $\widetilde{w}$ of w joins some lift of a curve in D to itself. Since every lift of a curve in D is separating, there is a sub-interval $w_0 \subset \widetilde{w}$ with both endpoints on the same side of some curve in the preimage of D. The image of such a sub-interval under the covering map is the desired wave. $\square$

Lemma 3.12. *Let C be a cut system and $\mathcal{I}$ a pattern. Suppose that $I \subsetneq J$ are two active intervals of $\mathcal{I}$, and that w is a wing of $I \subsetneq J$.*

i) If w is V-trivial, then there are directly nested active intervals $I' \subsetneq J'$ which have a wing $w' \subset w$ that defines a wave.

ii) Suppose now that $I \subsetneq J$ are directly nested, and that w defines a wave. Let C' be disjoint from C and let $\mathcal{I}'$ be a pattern which is compatible with $\mathcal{I}$. Then either the wing w can be made disjoint from $C'(\mathcal{I}')$, or there is a pair $I' \subsetneq J'$ of directly nested intervals in $\mathcal{I}'$ whose wing w' is a wave and contained in w.

Proof. i) Consider $I = I_1 \subsetneq I_2 \cdots \subsetneq I_k = J$ a maximal chain of nested final intervals in the pattern. Observe that, up to homotopy, all intersections of w with $C(\mathcal{I})$ correspond to endpoints of the I_j, since intersections with $J \subset I_k \setminus I_{k-1}$ can be removed up to homotopy (compare Figure 2). Since w is V-trivial loop, its intersection with $C(\mathcal{I})$ has a wave by Lemma 3.11. The endpoints of this wave then lie on the desired directly nested final intervals.

ii) Arguing as in i), we see that an essential intersection of the wing w with the system $C'(\mathcal{I}')$ can only happen if there are intervals $I \subsetneq I' \subsetneq J$ with $I' \in \mathcal{I}'$. In fact, the essential intersections of w exactly correspond to intervals $I'_j \in \mathcal{I}'$ with $I \subsetneq I'_1 \subsetneq \cdots \subsetneq I'_k \subsetneq J$. Since w defines a trivial loop in $\pi_1(V)$, a subarc $w_1 \subset w$ therefore defines a trivial loop with endpoints on $C'(\mathcal{I}')$. Applying part i) to w_1 then yields the desired wave w'.

$\square$

Finally, we choose once and for all a hyperbolic metric on the surface S. This allows us to talk about the length of intervals. The *length* of a pattern is the sum of the lengths of all intervals chosen by the pattern. We can now describe the *wave exchange move.* Suppose that $\mathcal{I}$ is a pattern for some cut system C, and suppose that $I \subsetneq J$ are directly nested active intervals whose wings w_1, w_2 are waves. The wave exchange $\mathcal{I}(w_1, w_2)$ is the set obtained by replacing $\{I, J\}$ by $\{w_1, w_2\}$.

Lemma 3.13. *The wave exchange $\mathcal{I}(w_1, w_2)$ is a pattern of strictly smaller length than $\mathcal{I}$.*

Proof. We begin by noting that $\mathcal{I}(w_1, w_2)$ satisfies property N) since the intervals $I \subsetneq J$ are supposed to be directly nested. Hence, any interval in $\mathcal{I}$ which intersects J is contained in $J - I$, and therefore nested inside w_1 or w_2. Property P) is obvious, since $\mathcal{I}$ and $\mathcal{I}(w_1, w_2)$ connect the same intersection points of δ with C. To show property F), note that $C(\mathcal{I}(w_1, w_2))$ is the full surgery of $C(\mathcal{I})$ at the wave w_1 (or w_2), and therefore is still filling (compare Figure 3). The claim about length is due to the fact that $w_1 \cup w_2 \subsetneq J$. $\square$

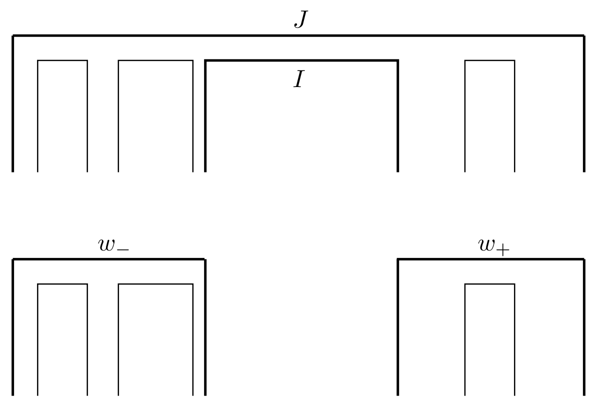

FIGURE 3. Suppose $I \subset J$ are directly nested, with wings defining waves, depicted above. Below is the wave exchange.

Definition 3.14. Call a pattern *essential* if no nested intervals define a V-trivial arc.

Corollary 3.15. *Any maximal length sequence of wave exchanges starting with a pattern $\mathcal{I}$ has finite length, and terminates in an essential pattern.*

Proof. The claim about finite length of wave exchange sequences follows since by the previous lemma each wave exchange strictly decreases the length of the pattern. Suppose that a pattern does not admit a wave exchange. Then by Lemma 3.12, no nested pair of intervals defines a V-trivial arc, as otherwise there would be a pair defining a wave, and hence the pattern would admit another wave exchange. □

Lemma 3.16. *Suppose that $\mathcal{I}$ is a pattern for some cut system C, and suppose that $I \subsetneq J$ are directly nested active intervals whose wings w_1, w_2 are waves. Suppose further that C' is disjoint from C and that $\mathcal{I}'$ is a compatible pattern.*

Then either $\mathcal{I}(w_1, w_2)$ is compatible with $\mathcal{I}'$, or there is a wave exchange $\mathcal{I}'(w_1', w_2')$ which is compatible with $\mathcal{I}$.

Proof. If w_1 (and thus w_2) can be made disjoint from $C'(\mathcal{I}')$, then the wave exchange $\mathcal{I}(w_1, w_2)$ is compatible with $\mathcal{I}'$. As in Lemma 3.12, this happens exactly if there is no $\mathcal{I}'$–interval nested between the $\mathcal{I}$–intervals defining w_1, w_2. Otherwise, by Lemma 3.12, there are wave-wings w_1', w_2' contained in w_1, w_2, defined by intervals $I', J' \in \mathcal{I}'$. Since we assumed that $I \subset J$ are directly nested, no interval in $\mathcal{I}$ nests between I' and J'. Hence, by reversing the roles, $\mathcal{I}'(w_1', w_2')$ is compatible with $\mathcal{I}$. □

Lemma 3.17. *Let C be a cut system, and $\mathcal{I}$ an essential pattern. Suppose that $I \subsetneq J$ are two active intervals, belonging to the same curve in $C(\mathcal{I})$.*

Let $\mathcal{I}'$ be a pattern for C' which is compatible with $\mathcal{I}$. Then there is an interval $K \in \mathcal{I}'$ so that $I \subset K \subset J$.

Proof. By Lemma 3.12 i), the wings of $I \subset J$ define nontrivial elements of $\pi_1(V)$. Hence, they cannot be disjoint from $C'(\mathcal{I}')$. This is only possible if there is an interval K as desired (again, non-nested intervals do not contribute intersections; compare Figure 2). $\square$

Corollary 3.18. *There is a number $D > 0$ with the following property.*

Suppose that C is a cut system, and $\mathcal{I}$ is an essential pattern for C. Let $\mathcal{I}'$ be a pattern for C' which is compatible with $\mathcal{I}$.

Suppose that $I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_k$ is a chain of intervals in $\mathcal{I}$ with $k \geq D$. Then there is an interval $K \in \mathcal{I}'$ so that $I_1 \subset K \subset I_k$.

Proof. Since a cut system has at most g curves, if D is large enough, there will be indices i, j so that I_i, I_j lie on the same curve of C_n. Then Lemma 3.17 applies and yields the desired interval. $\square$

3.3. **Proof of undistortion.** Before we can prove the main theorem, we need the following two results that connect patterns to usual subsurface projections.

Lemma 3.19. *Suppose that C is a cut system and that C' is a cut system which is disjoint from C. Then for any $K > 0$ there is a constant $L = L(K) > 0$ so that the following holds.*

Let $\mathcal{I}$ be an essential pattern for C, and let $\mathcal{I}'$ be a compatible, essential pattern for C'. Suppose that a is an arc which intersects $C(\mathcal{I})$ only in its endpoints, and so that $i(a, C') \leq K$. Then,

$$i(a, C'(\mathcal{I}')) < L$$

Proof. Any curve in $C'(\mathcal{I}')$ consists of arcs of C' and intervals in $\mathcal{I}'$. An arc in C' contributes at most K intersection points with a, while an interval in $\mathcal{I}'$ contributes at most two intersection points. Hence, to show the existence of the desired L, it suffices to show that the number of nested intervals in $\mathcal{I}'$ which do not contain a nested interval in $\mathcal{I}$ is bounded. This is exactly guaranteed by Corollary 3.18 $\square$

We are now ready to prove the main theorem. The core is the following lemma.

Lemma 3.20. *Suppose that C_n is a sequence of cut systems, so that consecutive C_i are disjoint. Then there is a sequence $\mathcal{I}_n$, so that for each n $\mathcal{I}_n$ is an essential pattern for C_n, and the patterns $\mathcal{I}_n, \mathcal{I}_{n+1}$ are compatible.*

Proof. As a first step, we apply Lemma 3.10 i) and ii) to obtain a sequence of surgery patterns $\mathcal{I}_n^1$ for C_n, so that for any n, the patterns $\mathcal{I}_n^1, \mathcal{I}_{n+1}^1$ are compatible. Next, we will inductively define sequences $\mathcal{I}_n^i$ of wave exchanges starting in $\mathcal{I}_n^1$. Let $m_i \geq 1$ be so that for any i, the patterns $\mathcal{I}_i^j$ are defined for $j \leq m_i$, and assume that for any n, the patterns $\mathcal{I}_n^{m_n}$ and $\mathcal{I}_{n+1}^{m_{n+1}}$ are compatible. Suppose that n is minimal with the property that $\mathcal{I}_n^{m_n}$ is not essential. In that case, consider a wave exchange $\mathcal{I}_n^{m_n+1}$ of $\mathcal{I}_n^{m_n}$. By minimality of n, the pattern $\mathcal{I}_{n-1}^{m_{n-1}}$ is essential and hence it admits no wave exchanges. Lemma 3.16 implies thus that $\mathcal{I}_{n-1}^{m_{n-1}}$ and $\mathcal{I}_n^{m_n+1}$ are still compatible. Similarly, Lemma 3.16 implies that either $\mathcal{I}_n^{m_n+1}$ is compatible with $\mathcal{I}_{n+1}^{m_{n+1}}$, or $\mathcal{I}_{n+1}^{m_{n+1}}$ admits a wave exchange $\mathcal{I}_{n+1}^{m_{n+1}+1}$ which is compatible with $\mathcal{I}_n^{m_n+1}$. By induction, we can therefore assume that m_n is replaced by m_n+1 (and possibly some $m_i, i > n$ are increased by one as well), keeping the other properties of the pattern. By Corollary 3.15, this procedure terminates in the desired sequence $\mathcal{I}_n$ of patterns. □

Now suppose that $(C_n, l_n), n = 1, \dots, N$ is a sequence in $\mathcal{G}$ connecting two points in $\mathcal{G}(\delta)$. Apply Lemma 3.19 to obtain a sequence $\mathcal{I}_n$ of patterns as in that lemma. For each n, consider now

$$A'_n = (\partial V - C_n(\mathcal{I}_n)) \cap l_n,$$

and choose for each n, a subset $A_n \subset A'_n$ containing a representative of each homotopy class of arc in A'_n. Define

$$\Gamma_n = C_n(\mathcal{I}_n) \cup A_n,$$

and note that each Γ_n defines a vertex of $\mathcal{R}(\delta)$. Observe that each arc $a \in A_n$ satisfies the prerequisites of Lemma 3.19, and there is therefore a number $K > 0$ so that $i(\Gamma_n, \Gamma_{n+1}) < K$ for all n. Hence, Γ_n defines a path in $\mathcal{R}(\delta)$ of length bounded above linearly by N.

Furthermore, since the only pattern for a cut system disjoint from δ is the empty pattern, this path connects the image of (C_1, l_1) and (C_N, l_N) under the map $U : \mathcal{G}(\delta) \to \mathcal{R}(\delta)$. Hence, distance between (C_1, l_1) and (C_N, l_N) is comparable to N, showing Theorem 3.1. □

Remark 3.21. *In order to prove the analogue of Theorem 3.1 for compression body groups, only two minor modifications are necessary. First, in Lemma 3.10, we have to do full surgeries, in order to keep the systems filling disc systems for the compression body. This is possible by Lemma 2.1. Second, the model $\mathcal{G}$ needs to be adapted to compression bodies so that the loop l is filling together with the filling disc system (the model $\mathcal{R}$ need not be modified).*

4. Primitive and Annulus Stabilisers

In this section we encounter distorted stabilisers in the handlebody group. There will be two classes of such stabilisers that we consider – those of primitive curves, and those of primitive annuli.

Recall that a curve α on the boundary of a handlebody is called *primitive* if it defines a primitive element in the (free) fundamental group $\pi_1(V)$. Equivalently, α is primitive if there is a meridian δ which intersects α in a single point.

A *primitive annulus* will mean a pair α_1, α_2 both of which are primitive, and which bound an embedded annulus A.

Before beginning the discussion in earnest, we will first summarise the results that are proven in this section.

Theorem 4.1. *Let $V_g, g \geq 3$ be a handlebody of genus at least three, and let $\alpha \subset V_g$ be primitive. Then the stabiliser of α is exponentially distorted.*

The genus requirement in Theorem 4.1 is necessary, as the following proposition shows.

Proposition 4.2. *Let V_2 be a handlebody of genus 2, and let α be primitive. Then the stabiliser of α is undistorted.*

In fact, there are more distorted stabilisers in the handlebody group:

Theorem 4.3. *Let V_g be a handlebody, and suppose that A is an annulus whose boundary consists of primitive elements. Suppose that $g \geq 3$ if A is non-separating, or that $g \geq 4$ if A is separating. Then the stabiliser of A is exponentially distorted.*

Remark 4.4. *It is not clear if the genus bound in Theorem 4.3 is optimal.*

To the knowledge of the author, the analogous statement of Theorem 4.3 for $\mathrm{Out}(F_n)$ is new as well:

Proposition 4.5. *The stabiliser of a cyclic splitting in $\mathrm{Out}(F_n)$ is exponentially distorted.*

We also note that the proofs of upper distortion bounds also show that the stabilisers in question are finitely generated.

4.1. **Algebraic description of primitive stabilisers.** Stabilisers of primitive curves, in contrast to the situation of meridians, cannot easily be reduced to lower-genus handlebody groups and point-pushing. In this subsection we discuss some of the difficulties encountered when trying to extend the usual description of stabilisers using boundary pushing and a reduction to smaller genus as for the mapping class group of a surface. A reader interested only in the geometry of stabilisers may safely skip to the next subsection.

Throughout, α will be a primitive loop on ∂V. In particular, α is non-separating. Let

$$\partial V - \alpha = Y,$$

where Y has two boundary components α_+, α_- corresponding to the two sides of α. Recall that in the mapping class group of ∂V, we have a short exact sequence

$$1 \to \mathbb{Z} \to \mathrm{Mcg}(Y) \to \mathrm{Stab}_{\mathrm{Mcg}}(\alpha) \to 1 \tag{1}$$

where the first map sends 1 to $T_{\alpha_+}T_{\alpha_-}^{-1}$. Let $\hat{Y}$ be the surface gluing a disc to the boundary component α_- of Y. We also have a Birman exact sequence

$$1 \to \pi_1(U\hat{Y}) \to \mathrm{Mcg}(Y) \to \mathrm{Mcg}(\hat{Y}) \to 1, \tag{2}$$

where $U\hat{Y}$ denotes the unit tangent bundle of $\hat{Y}$. This sequence splits, for example in the following way: we define a *α–splitting surface* to be a subsurface S of Y so that $Y - S$ is a 3–holed sphere containing α_-, α_+ in its boundary. Then the inclusion $\mathrm{Mcg}(S) \to \mathrm{Mcg}(Y)$ yields the desired splitting. Hence, we can identify $\mathrm{Mcg}(Y) \cong \pi_1(U\hat{Y}) \rtimes \mathrm{Mcg}(\hat{Y})$.

Since α is primitive, neither Y nor $\hat{Y}$ can be naturally identified with a sub-handlebody. However, we can choose a α–splitting surface S which is the boundary of a sub-handlebody, and use the splitting of the sequence (2) to identify with quotient with $\mathrm{Mcg}(S)$. We then obtain

$$1 \to \pi_1(U\hat{Y}) \cap \mathcal{H}(V) \to \mathrm{Mcg}(Y) \cap \mathcal{H}(V) \to \mathcal{H}(S) \to 1$$

Therefore, describing the stabiliser of α relies on describing the subgroup $\Gamma = \pi_1(U\hat{Y}) \cap \mathcal{H}(V)$ of the boundary pushing subgroup.

Recall that we have a short exact sequence

$$1 \to \mathbb{Z} \to \pi_1(U\hat{Y}) \to \pi_1(S) \to 1,$$

and under the identification of $\pi_1(U\hat{Y})$ with a subgroup of $\mathrm{Mcg}(Y)$, the kernel corresponds to the Dehn twist about α. In particular, since the twist about α is not contained in the handlebody group of V, the group Γ intersects each fibre of $\pi_1(U\hat{Y}) \to \pi_1(S)$ in at most one point.

Intuitively, it is clear that the group Γ is much smaller than $\pi_1(S)$. Namely, consider a meridian δ which intersects α in a single point. If $a \subset Y$ is an arc based at α_- disjoint from δ except in its endpoints, then the push about a maps δ to the curve obtained by concatenating $\delta \cap Y$ with a. Hence, in order for the push to be in $\mathcal{H}(V)$, the arc a would have to define a meridian as well. In fact, as the following lemma shows, Γ can be generated by such elements.

Lemma 4.6. *The intersection of $\pi_1(T^1S)$ with the handlebody group is generated by the image of all loops in $S - \alpha$ which are embedded meridians. These elements correspond to annulus twists about annuli one of whose boundary components is α, composed with meridian twists.*

Before proving Lemma 4.6, we want to mention that although pushes about embedded meridians generate Γ, the group does not simply consist of pushes along V–trivial arcs. In fact, we have the following.

Lemma 4.7. *For V of genus $g \geq 3$, the group Γ is not normal in $\pi_1(U\hat{Y})$.*

We prove Lemma 4.7 in the appendix, since the proof only consists of a careful, somewhat lengthy check of intersection patterns. However, we want to emphasise the following consequence of Lemma 4.7 in combination with Lemma 4.6, which may be of independent interest, and highlights another difference between the complements of meridians and primitives in a handlebody.

Corollary 4.8. *The kernel $\ker(\pi_1(S - \alpha) \to \pi_1(V))$ of the map induced by inclusion is not generated by embedded curves.*

Proof of Lemma 4.6. Recall that $\Gamma = \pi_1(U\hat{Y}) \cap \mathcal{H}$, and denote by Γ_0 the subgroup generated by the pushes as in the statement of the lemma. Pick a point $p \in \alpha$, and note that it defines points $p_-, p_+ \in Y$. Define $\mathcal{A}$ to be the graph whose vertices correspond to arcs $a \subset Y$ joining p_- to p_+, so that a defines a meridian on ∂V. We join two vertices with an edge, if the corresponding arcs are disjoint except at their endpoints. Note that $\mathcal{H}(V) \cap \mathrm{Mcg}(Y)$ acts on $\mathcal{A}$ as isometries.

Observe that there is an arc a as above, so that $Y \setminus a$ is homotopy equivalent to the splitting surface S. Hence, the stabiliser of this arc a in $\mathcal{H}(V) \cap \mathrm{Mcg}(Y)$ is equal to (the image of) $\mathcal{H}(S)$.

To prove the lemma, it therefore suffices to show that Γ_0 acts transitively on the vertex set of $\mathcal{A}$. To this end, first consider two arcs a, a' joined by an edge. Then the concatenation $l = a^{-1} * a'$ is a loop joining p_- to itself, and furthermore it defines an embedded meridian in Y. Hence, the push about l is an element of the handlebody group (it is an annular twist). Furthermore, we have

$$P(l)(a') = a.$$

Hence, to prove the claim, it suffices to show that $\mathcal{A}$ is connected. This follows from a standard surgery argument: suppose a, a' are any two arcs representing vertices that are not disjoint. Since they both define meridians on ∂V, there is a wave $w \subset a'$. A suitable surgery a_w then intersects Y in an arc still connecting p_- to p_+, which is otherwise disjoint from a, and intersects a' in strictly fewer points. □

4.2. **Lower distortion bounds.** The proofs of the lower distortion bounds for all three results mentioned at the beginning of this section are very similar, and rely on two main ingredients. On the one hand, we use the following theorem, which is shown by Handel-Mosher ([HM, Section 4.3, Case 1]):

Theorem 4.9. *Let $n \geq 3$ be given, and F_n is a free group with free basis $e_1, \dots, e_n$. Suppose that $\Theta : \langle e_1, e_2 \rangle \to \langle e_1, e_2 \rangle$ is an irreducible automorphism of exponential growth. Define an automorphism $f_k \in \mathrm{Out}(F_n)$ by the rule*

$$\begin{aligned} e_i &\mapsto e_i, \quad i < n \\ e_n &\mapsto e_n \Theta^k(e_1). \end{aligned}$$

Then the norm of f_k in the stabiliser of the conjugacy class $[e_1]$ grows exponentially in k.

The second ingredient is a construction similar to the one employed in Section 3 of [HH2].

Namely, let X be a surface of genus 1 with one boundary component. Consider the 3–manifold $W = X \times [0, 1]$, which is a handlebody of genus 2. The boundary

$$\partial W = X^0 \cup A \cup X^1, \quad X^i = \{i\} \times X, A = \partial X \times [0, 1]$$

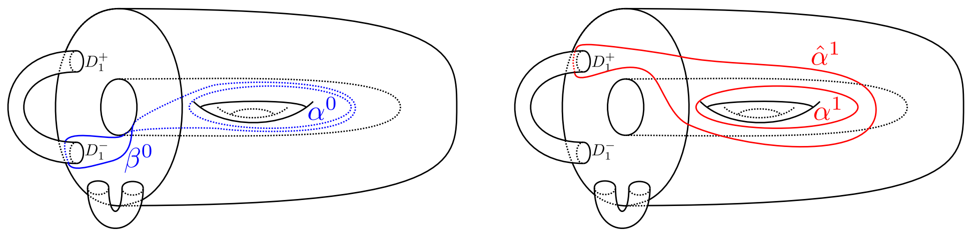

FIGURE 4. Left: The setup to construct distorted curve stabilisers. Right: A non-separating annulus fixed by the elements f_k. A separating annulus with the same property could be constructed by making $\hat{\alpha}^1$ surround the lower handle instead.

consists of two copies of X and an annulus A.

Choose $2(g-2)$ disjoint disks $D_i^-, D_i^+ \subset \mathrm{int}(A)$, for $i = 3, \ldots, g$ and for each i attach a three-dimensional 1-handle h_i to D_i^+, D_i^- to obtain a handlebody V of genus g.

Let $\alpha \subset X$ be a non-separating simple closed curve, and denote by $\alpha^j = \alpha \times \{j\}$ for $j = 0, 1$. Observe that α^0, α^1 are homotopic in V, and are primitive. Let $\beta^0 \subset (A \setminus \cup_i(D_i^+ \cup D_i^+)) \cup X^0$ be a simple closed curve which bounds a pair of pants together with ∂D_1^- and α^0 (compare Figure 4). Consider the mapping class

$$P = T_{\alpha^0} T_{\beta^0}^{-1} \in \mathcal{H}(V),$$

which is a handle slide.

Choose a basis $e_1, \ldots, e_g$ for $\pi_1(V)$, so that e_1, e_2 correspond to loops in X^0, the loop α^0 defines the conjugacy class of e_1, and the loops $e_3, \ldots, e_g$ are dual to the handles h_i, not entering $X^0 \cup X^1$. We summarise some important properties of P in the following lemma.

Lemma 4.10. *The element P induces the following automorphism on $\pi_1(V)$ with respect to the basis chosen above:*

$$e_i \mapsto e_i, i < g$$

$$e_g \mapsto e_g e_1.$$

Furthermore, P fixes the curve α^0 and restricts to the identity on X^1.

Let ψ be a pseudo-Anosov element of X which induces an irreducible, exponentially growing automorphism Θ of $\pi_1(X) = F_2$. Define the mapping classes

$$f_k = \psi^k P \psi^{-k}.$$

Observe that each f_k lies in the handlebody group, and we have the following

Lemma 4.11. *The element f_k induces the following automorphism on $\pi_1(V)$ with respect to the basis chosen above:*

$$e_i \mapsto e_i, i < g$$

$$e_g \mapsto e_g \Theta^k(e_1).$$

Furthermore, each f_k fixes

i) the curve α^1,
ii) a non-separating annulus A, one boundary component of which is α^1,
iii) for each $h = 1, \ldots, g-2$, a separating annulus A_h, one boundary component of which is α^1, and whose complement has genus h.

Here, the annuli A, A_h do not depend on k.

Proof. The claim on the action on fundamental group is clear from Lemma 4.10. Also observe that since P acts as the identity on X^1, the same is true for f_k. This immediately implies that the mapping class f_k preserves the loop $\alpha^1 \subset X^1$. Furthermore, we can choose a curve $\hat{\alpha}^1$ which is disjoint from α^0, β^0 and bounds an annulus A together with α^1. By choosing the $\partial D_i^\pm$ to lie on the correct side of $\hat{\alpha}^1, \alpha^1$. In this way, we can ensure that the annulus A can be non-separating or separating, and in the latter case, we can choose the genus separated off freely between 1 and $g-2$. □

Now we are ready to prove the lower distortion parts of the theorems mentioned at the beginning of this section.

Proof of the lower bound in Theorem 4.1. The stabiliser of any primitive curve α is conjugate, in the handlebody group, to the stabiliser of α^1. Hence, it suffices to show that the stabiliser of α^1 is at least exponentially distorted. We use the elements f_k as above. Namely, we have a Lipschitz map

$$\pi : \mathrm{Stab}_{\mathcal{H}(V)}(\alpha^1) \to \mathrm{Stab}_{\mathrm{Out}(F_g)}([e_1]),$$

since α^1 and α^0 define the conjugacy class $[e_1]$ in $\pi_1(V)$. By Theorem 4.9, the elements $\pi(f_k)$ have norm growing exponentially in k. Hence, the same is true for $f_k \in \mathrm{Stab}_{\mathcal{H}(V)}(\alpha^1)$. On the other hand, as $f_k = \psi^k P \psi^k$, the norm of f_k in $\mathcal{H}(V)$ is clearly growing linearly in k. This shows that $\mathrm{Stab}_{\mathcal{H}(V)}(\alpha^1)$ is at least exponentially distorted in $\mathcal{H}(V)$. □

Proof of the lower bound in Theorem 4.3. The stabiliser of any annulus as in that theorem is conjugate to an annulus A or A_h as in Lemma 4.11. Now, we can finish the proof just like the previous argument. Namely, the elements f_k as above fix A, A_h, and also the stabilisers of these annuli are contained in the stabiliser of α^1. □

As mentionend in the introduction, the lower distortion bound in Proposition 4.5 follows directly from [HM]. For completeness, we include a proof (from a topological perspective).

Proof of the lower bound in Proposition 4.5. We use the connection of $\mathrm{Out}(F_n)$ to the mapping class group of a the double of a handlebody. Let W be the closed 3–manifold obtained by doubling V about its boundary. Recall the short exact sequence [Lau, Théorème 4.3, Remarque 1)]

$$1 \to K \to \mathrm{Mcg}(W) \to \mathrm{Out}(F_n) \to 1$$

where K is finite, and the right map is induced by the action on the fundamental group. We also have a natural map $\mathcal{H} \to \mathrm{Mcg}(W)$ obtained by doubling, so that the composition $\mathcal{H} \to \mathrm{Mcg}(W) \to \mathrm{Out}(F_n)$ agrees with the action on the fundamental group of the handlebody.

Under the doubling map $\mathcal{H} \to \mathrm{Mcg}(W)$, the stabiliser of an annulus A as above in $\mathcal{H}$ maps to the stabiliser of a torus T_A, so that the image of $\pi_1(T_A)$ in $\pi_1(W)$ is generated by $[e_1]$. Under the map $\mathrm{Mcg}(W) \to \mathrm{Out}(F_n)$ the stabiliser of T_A maps to the stabiliser of a cyclic splitting Z, where the amalgamating group is generated by $[e_1]$. The same argument as above then shows that the image of the sequence f_k has length growing exponentially in k in the stabiliser of Z. □

4.3. **Upper distortion bounds.** The upper distortion bounds in Theorem 4.1, 4.3 and Proposition 4.5 follow from a surgery construction. We begin by describing the case of an annulus A in a handlebody in detail; the case of a primitive element is very similar. Then we discuss the case of a cyclic splitting in the free group.

We are again using the two complexes $\mathcal{G}$ and $\mathcal{R}$ which appeared in Section 3. In fact, we consider the following sub-complex

Definition 4.12. Let $\mathcal{G}(A)$ to be the full sub-complex of $\mathcal{G}$ of all those vertices whose cut system C intersects each curve in ∂A in exactly one point, and also so that l intersects each curve in A at most in one point.

Definition 4.13. Let $\mathcal{R}(A)$ be the full sub-complex of $\mathcal{G}$ of all those vertices whose cut system C intersects each curve in ∂A in exactly one point, and also so that ∂A embeds in the graph as an embedded subgraph.

By choosing the constants larger, we can make $\mathcal{G}(A)$ and $\mathcal{R}(A)$ connected, and therefore quasi-isometric to the stabiliser of A in the handlebody group.

Lemma 4.14. *Suppose that C, C' are cut systems both of which intersect A minimally. Then there is a surgery C_1 of C in direction of C' which also intersects A minimally.*

Proof. Let α be one of the boundary component of A. Since α intersects C' in a single point, and C' defines at least two distinct waves with respect to C, there is a wave w which does not intersect α. Let C_1 be the cut system surgery defined by that wave w. As the wave w is disjoint from α, the result C_1 intersects α in at most one point. As α is nontrivial in $\pi_1(V)$, it cannot be disjoint from a cut system – hence, α intersects C_1 in a single point.

Consider now the second boundary component β of A. Since β intersects both C and C' in one point, it intersects C_1 in at most two points. However, β is freely homotopic to α, and therefore has to intersect C_1 also in a single point. The case of zero intersections is impossible by the same argument as above. The case of two intersection points is impossible since then it would either be trivial, or a reduced word of length two, which cannot be conjugate to a reduced word of length one. □

As a consequence, we get

Corollary 4.15. *Given two cut systems C, C', both of which intersect A minimally, there is a sequence*

$$C = C_0, C_1, \ldots, C_n, C_{n+1} = C'$$

so that all C_i intersect A minimally, and $n \leq i(C, C')$.

We also note the following, which follows e.g. from [HH1, Corollary A.4]:

Proposition 4.16. *There are numbers a, b so that the following holds. Let C be any multicurve, and $f \in \mathcal{H}(V)$ be a handlebody group element. Then*

$$i(C, f(C)) \leq a \cdot b^{\|f\|_{\mathcal{H}(V)}}.$$

Now fix a point $(C, l) \in \mathcal{G}(A)$, and an element $f \in \mathrm{Stab}(A)$. We then know, from Proposition 4.16 that

$$i(C \cup l, f(C \cup l)) < a \cdot b^{\|f\|_{\mathcal{H}(V)}} = M$$

Let $\Gamma \subset f(C) \cup f(l)$ be the vertex of $\mathcal{R}$ corresponding to $(f(C), f(l))$, and let C_i be the surgery sequence guaranteed by Corollary 4.15. We now put $\Gamma_0 = \Gamma$ and define a sequence in $\mathcal{R}$ by inductively applying the following two lemmas. For their formulation, suppose that Γ is a graph representing a vertex of $\mathcal{R}$. Recall that this means that in particular, there is an embedded cut system $C \subset \Gamma$. We denote this system by $C(\Gamma)$, and we call any edge of Γ which is not contained in C a *rope edge*.

Lemma 4.17. *There is a constant $L_1 > 0$ with the following property. Suppose that Γ_i is a vertex of $\mathcal{R}$, so that each rope edge e of Γ_i intersects C in at most K points, and each rope edge is disjoint from A. Then there is a vertex Γ_i' of $\mathcal{R}$ with the following properties:*

i) $C(\Gamma_i) = C(\Gamma_i')$.
ii) *Each rope edge of Γ_i' is disjoint from every A-arc.*
iii) *Each arc in $C \cap (\partial V - C(\Gamma_i'))$ is disjoint from the rope edges of Γ_i' up to homotopy.*
iv) *Each curve of C disjoint from $C(\Gamma_i)$ is embedded in Γ_i'.*
v) *The distance between Γ_i and Γ_i' in $\mathcal{R}(A)$ is at most $L_1 K$.*

Proof. To obtain Γ_i' from Γ_i, surger each rope edge in the direction of all C-arcs or disjoint curve in C. Since C intersects each curve of A in a single point, we can choose these surgeries so that the rope edges stay disjoint from the A-arcs.

For any pair of a C–arc and rope edge, at most K surgeries are needed to make them disjoint, and each surgery step stays in $\mathcal{R}(A)$. Since the number of different C–arcs is uniformly bounded by the genus of ∂V, and the same is true for the rope edges of Γ_i, this shows the lemma. □

Lemma 4.18. *There is a constant L_2 with the following property. Suppose that Γ_i' is a vertex of $\mathcal{R}$, so that each rope edge of Γ_i' is disjoint from C.*

Further, suppose that C_{i+1} is a cut system obtained from $C_i = C(\Gamma_i)$ from a surgery move in the direction of C.

Then there is a vertex Γ_{i+1} of $\mathcal{R}$ with the following properties:

i) $C(\Gamma_{i+1}) = C_{i+1}$.
ii) Each rope edge of Γ'_i is disjoint from every A-arc.
iii) Every rope edge of Γ_{i+1} intersects C in at most $i(C, C_i)$ points.
iv) The distance between Γ_i, Γ_{i+1} is at most L_2.

Proof. Let w be the wave defining the surgery. We may replace Γ_i by a vertex so that w is a rope edge. This does not violate ii). Then, Γ_i contains C_{i+1} as a subgraph, and we define Γ_{i+1} to be this vertex, guaranteeing i). Any rope edge of C_{i+1} is now either a rope edge of Γ_i, or a subarc of C_i. hence, property iii) holds. Property iv) is clear. □

Proof of Theorem 4.3. Observe that the distance between Γ_i and Γ_{i+1} in $\mathcal{R}(A)$ can be bounded by $L_1 M + L_2$ for any i. Hence, the distance between Γ and Γ_N in $\mathcal{R}(A)$ is at most $M(L_1 M + L_2)$. By the triangle inequality, the distance between $U(C, l)$ and Γ_N in $\mathcal{R}$ is therefore at most $M(L_1M+L_2)+M$. However, since $U(C, l)$ and Γ_N share the same cut system C as an embedded graph, and the stabiliser of $C \cup A$ is undistorted in the handlebody group, we actually obtain that the distance between $U(C, l)$ and Γ_N is at most $L_3(M(L_1M+L_2)+M)$ in $\mathcal{R}(A)$ for some uniform constant L_3. This implies by the triangle inequality, that the distance between $U(C, l)$ and Γ is at most $(L_3+1)(M(L_1M+L_2)+M)$. Since this is polynomial in M, it is exponential in $\|f\|_{\mathcal{H}(V)}$, showing that $\mathcal{R}(A)$ is at most exponentially distorted. This shows the upper bound in Theorem 4.3. □

To prove Proposition 4.5, we work in a doubled handlebody. First, using surgeries of sphere systems instead of disc systems we show the following analogue of Corollary 4.15 with essentially the same argument.

Corollary 4.19. *Let $W = \#_g S^1 \times S^2$ is a doubled handlebody, and suppose that $T \subset W$ is an embedded torus so that the image of $\pi_1(T) \to \pi_1(W)$ is a cyclic group generated by a primitive element.*

Suppose that σ, σ' are two sphere systems in minimal position, each of which intersects T in a single circle. Then there is a sequence

$$\sigma = \sigma_0, \sigma_1, \dots, \sigma_n, \sigma_{n+1} = \sigma'$$

so that each σ_i intersects T in a single circle, and n is at most the number of intersection circles in $\sigma \cap \sigma'$.

We also have the following

Lemma 4.20. *Let $W = \#_g S^1 \times S^2$ be a doubled handlebody. Then there are numbers a, b so that the following holds. Let σ be any filling sphere system, and $f \in \mathrm{Mcg}(W)$ be arbitrary. Then, in minimal position, the number of intersection circles in $\sigma \cap f(\sigma)$ is at most*

$$a \cdot b^{\|f\|_{\mathrm{Mcg}(W)}}.$$

Proof. To show the lemma, it suffices to show that there is a number C so that if σ, σ' are any two sphere systems, and σ'' is disjoint from σ', then

$$i(\sigma, \sigma'') \leq Ci(\sigma, \sigma') + C$$

This follows since σ'' intersects each component of $\sigma \cap (W - \sigma')$ in at most one circle. □

Together with Corollary 4.19, this lemma proves the upper bound in Proposition 4.5 by induction.

Finally, we prove the undistortion statement in genus 2 for primitive stabilisers.

Proof of Proposition 4.2. As in Section 3, we aim to project paths in $\mathcal{R}$ to $\mathcal{R}(\alpha)$. First, we observe the following preliminary step. Suppose that $\Delta = \{\delta_1, \delta_2\}$ is any cut system. Then, since α is primitive, at least one of $\iota(\alpha, \delta_1), \iota(\alpha, \delta_2)$ is odd. In particular, there is a subarc $d \subset \delta_1 \cup \delta_2$ connecting the two different sides of α. One component of a regular neighbourhood of $d \cup \alpha$ is a separating meridian $\delta(d)$. As V_2 has genus 2, there is a unique cut system $\Delta(d)$ disjoint from $\delta(d)$. Since α is disjoint from $\delta(d)$, it intersects this cut system in a single point. Observe that if d' is any other possible choice of arc, the meridians $\delta(d), \delta(d')$ intersect in at most four points, and thus $\Delta(d), \Delta(d')$ also intersect in uniformly few points.

The same argument shows that if Δ' is a disjoint cut system, and d' is an admissible arc, then $\Delta(d), \Delta'(d')$ intersect in uniformly few points. Hence, we can define a Lipschitz projection of $\mathcal{H}_2$ to the stabiliser of α. □

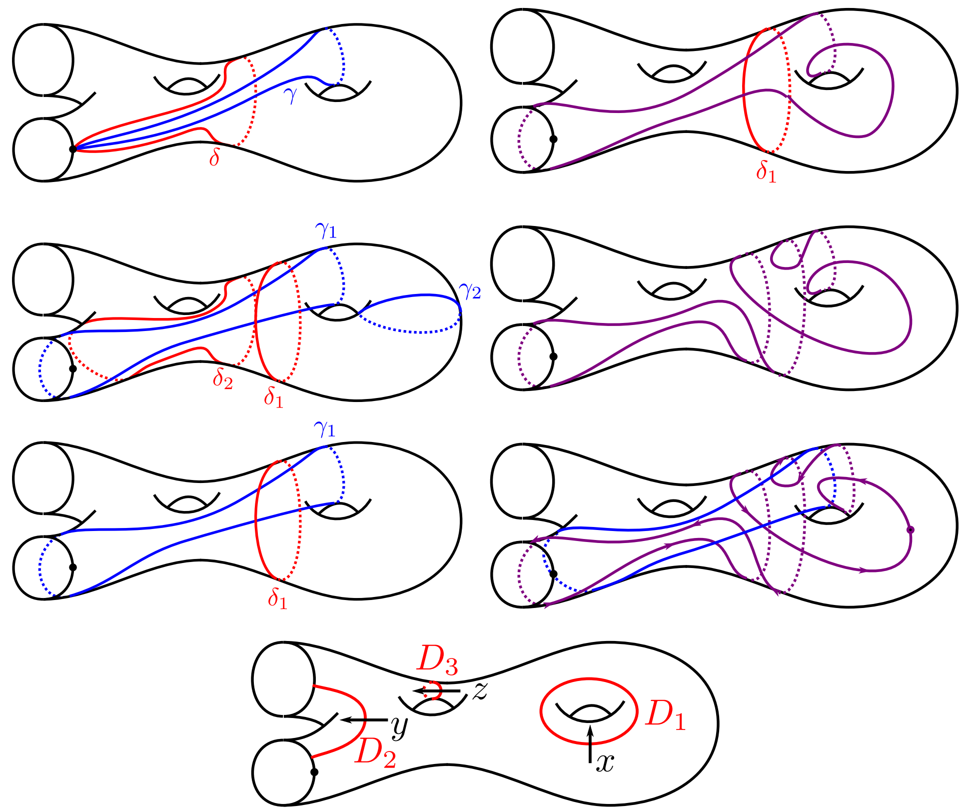

FIGURE 5. The left three pictures show the relevant curves in the proof of Lemma 4.7. For ease of depiction, in all of these pictures the handlebody structure is the “outside” handlebody in the standard Heegaard splitting of S^3. The three pictures on the right depict the action of the twist product on μ: the top shows $T_{\gamma_1}\mu = T_{\gamma_1}T_{\delta_1}\mu$, the middle one shows $T_{\delta_1}^{-1}T_{\gamma_1}T_{\delta_1}\mu$ and in the bottom one γ_1 is shown superimposed. Below is the basis used to compute the element after applying all twists.

APPENDIX A. THE PROOF OF LEMMA 4.7

We give the proof in the case of a genus 3 handlebody, but the method extends to any genus ≥ 3.

Consider two disjoint loops γ, δ as in Figure 5 on the left, based at the curve α, i.e. loops so that under the identification of $\hat{Y}$ with the splitting surface S, the loop δ is a meridian, while γ is a primitive curve. We will show that the boundary push about any element in the fibre of $\pi_1(U\hat{Y}) \to \pi_1(S)$ over the commutator $[\delta, \gamma]$ is not contained in the handlebody group. This is enough to prove the lemma.

A push along δ will be of the form

$$P_\delta = T_{\delta_1}^{-1} T_{\delta_2} T_\alpha^k$$

for some k, where δ_1, δ_2 are disjoint from δ and bound a pair of pants together with α. Similarly, a push along γ will be of the form

$$T_{\gamma_1}^{-1} T_{\gamma_2} T_\alpha^l$$

for some l, and γ_1, γ_2 disjoint and bounding a pair of pants with α. Since δ_2, γ_2 are disjoint from all other involved curves, and the corresponding Dehn twists therefore commute with all others involved in the definition of P_δ, P_γ, we can compute the commutator of the pushes as

$$\Psi = [P_\gamma, P_\delta] = T_{\gamma_1}^{-1} T_{\delta_1}^{-1} T_{\gamma_1} T_{\delta_1}.$$

To prove the lemma, we therefore need to show that $T_\alpha^n \Psi$ is not in the handlebody group for any n. To show this claim, we will study the effect of $T_\alpha^n \Psi$ on a meridian μ, which is disjoint from δ_1 and intersects γ_2 in a single point. First note that μ itself, as well as δ_1, γ_1 are disjoint from α, and so the image $T_\alpha^n \Psi(\mu)$ will not depend on n. Thus, we may assume $n = 0$.

The action of the first three twists is shown in Figure 5 on the right. Twists are executed right-to-left, T_x is a left-handed twist about the curve x.

Instead of actually performing the final twist, we can now determine the resulting word in $\pi_1(V)$ by recording intersections with a cut system as follows. We choose a cut system consisting of three disks D_1, D_2, D_3. Here, D_1 is freely homotopic to μ, D_2 intersects α in a single point, and D_3 is disjoint from all curves involved. Compare the bottom picture in Figure 5. We also choose transverse orientations, so that the cut system defines an oriented basis x, y, z of $\pi_1(V)$.

To find the element which $T_{\gamma_1}^{-1} T_{\delta_1}^{-1} T_{\gamma_1} T_{\delta_1} \mu$ defines in $\pi_1(V)$, we now follow along the purple curve, starting at the solidly drawn basepoint, turn right and follow γ_1 whenever we encounter γ_1, and keep track of intersections with D_1, D_2. With the transverse orientations as in Figure 5, this yields the following word (for readability, intersections due to γ_1 are bracketed):

$$x(x^{-1}y^{-1})(xy)y(y^{-1}x^{-1})(yx)(y^{-1}x^{-1}) = y^{-1}xyx^{-1}yxy^{-1}x^{-1}$$

This is a nontrivial element in $\pi_1(V)$, and therefore $T_{\gamma_1}^{-1} T_{\delta_1}^{-1} T_{\gamma_1} T_{\delta_1} \mu$ is not a meridian.

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