Small exercises

1. Let k be a field. Consider the ring
$$A = \left(\frac{k[x,y]}{(xy)}\right)_{(x,y)} .$$
Compute the rings $\mathcal{O}(U)$ for all open subsets $U \subset \operatorname{Spec} A$.

2. a) What are the possible residue fields of points in $\operatorname{Spec} \mathbb{R}[x]$?
   b) Describe the map
   $$\operatorname{Spec} \mathbb{C}[x] \to \operatorname{Spec} \mathbb{R}[x]$$
   induced by the field extension $\mathbb{R} \subset \mathbb{C}$.

3. Let A be a ring. Show that $\operatorname{Spec} A$ is compact.

4. Does there exist an affine scheme having exactly 1 closed and 3 non-closed points?

5. Consider the continuous map
$$f^*\colon \operatorname{Spec} B \to \operatorname{Spec} A$$
induced by a ring homomorphism $f\colon A \to B$.
   a) Suppose that f is surjective. Show that f^* is injective and its image is closed in $\operatorname{Spec} A$.
   b) Suppose that f is injective. Show that the image of f^* is dense in $\operatorname{Spec} A$ and give an example where f^* is not surjective.
   c) Suppose that $f^*\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is a homeomorphism. Does f need to be injective or surjective?

6. Let A be a ring and let $p \in \operatorname{Spec} A$. Show that $\{p\}$ is closed if and only if the map $A \to \kappa(p)$ is surjective.

7. For a subset $S \subset \operatorname{Spec} A$, let
$$I(S) = \{f \in A \mid S \subset V(f)\} .$$
   a) Show that $V(I(S)) = \overline{S}$ for any subset $S \subset \operatorname{Spec} A$.
   b) Show that $I(V(J)) = \sqrt{J}$ for any ideal $J \subset A$.

8. Let k be a field endowed with the discrete topology. Let A be the ring of all convergent sequences in k. Show that $\operatorname{Spec} A$ is homeomorphic to a closed subset of $\mathbb{R}$ and its structure sheaf $\mathcal{O}$ is precisely the sheaf of continuous functions to k.