Tutorial 11

Let k be a field.

1. One of the following categories is not abelian. Can you find it?

- ☐ The category of countably generated $k[x]$-modules
- ☐ The category of locally free $\mathcal{O}_{\mathbb{P}^1_k}$-modules
- ☐ The category of finite abelian groups whose order is a power of 3
- ☐ The category $\mathcal{C}$ defined by $\mathrm{ob}(\mathcal{C}) = \mathbb{N}_0$ and $\mathrm{Hom}_{\mathcal{C}}(m, n) = k^{n\times m}$ for all $m, n \in \mathbb{N}_0$, where the composition is given by matrix multiplication
- ☐ The category $\mathcal{A}^{\mathrm{op}}$ obtained from an abelian category $\mathcal{A}$ by reversing the directions of all arrows

2. Let $f\colon A \to B$ be a morphism in an abelian category.

a) Show that f is a monomorphism if and only if $\ker f = 0$.

b) Show that f is an epimorphism if and only if $\operatorname{coker} f = 0$.

c) Show that f is an isomorphism if and only if $\ker f = 0$ and $\operatorname{coker} f = 0$.

d) Show that there exists an isomorphism $\operatorname{coker}\ker f \cong \ker\operatorname{coker} f$ such that the following diagram commutes:

$$\begin{tikzcd}
\ker f \arrow[r, "\ker f"] \arrow[dr, "0"'] & A \arrow[rr, "f"] \arrow[d] & & B \arrow[r, "\operatorname{coker} f"] & \operatorname{coker} f \\
& \operatorname{coker}\ker f \arrow[rr, "\cong"] & & \ker\operatorname{coker} f \arrow[u] \arrow[ur, "0"'] &
\end{tikzcd}$$

3. For an abelian group A, we define $\mathrm{Ext}^i(A, -)$ as the i-th right derived functor of the left-exact functor $\mathrm{Hom}(A, -)$ from the category of abelian groups to itself. Compute the following abelian groups for all i:

$$\mathrm{Ext}^i(\mathbb{Z}/4, \mathbb{Z}/6) \qquad \mathrm{Ext}^i(\mathbb{Q}, \mathbb{Z}/5)$$

4. Let $f\colon A^\bullet \to B^\bullet$ be a morphism of complexes in an abelian category. We define the complex $C^\bullet(f)$ by $C^k(f) = A^{k+1} \oplus B^k$ for all $k \in \mathbb{Z}$, where the differential $d^{k-1}\colon C^{k-1}(f) \to C^k(f)$ is given as the sum of the three morphisms shown below:

$$\begin{tikzcd}
& A^k \arrow[r, "-d_A^k"] \arrow[dr, "f^k"] & A^{k+1} & \\
C^{k-1}(f) = & \oplus & \oplus & = C^k(f) \\
& B^{k-1} \arrow[r, "d_B^{k-1}"'] & B^k &
\end{tikzcd}$$

Show that f is a quasi-isomorphism if and only if $C^\bullet(f)$ has no cohomology.