Solutions to sheet 2

1. Let A be a ring and let $\mathcal{O}$ be the structure sheaf on $\operatorname{Spec} A$.

   a) Show that the stalk of $\mathcal{O}$ at $\mathfrak{p}$ is isomorphic to $A_\mathfrak{p}$.

   Proof. For any open neighbourhood U of $\mathfrak{p}$, consider the ring homomorphism

$$\begin{aligned} \mathcal{O}(U) &\to A_\mathfrak{p} \\ s &\mapsto s(\mathfrak{p}) \ . \end{aligned}$$

   Since these morphisms are compatible with the restriction maps, the universal property of the direct limit yields a well-defined ring homomorphism

$$\begin{aligned} \mathcal{O}_\mathfrak{p} &\to A_\mathfrak{p} \\ [(s,U)] &\mapsto s(\mathfrak{p}) \ . \end{aligned}$$

   We aim to show that this map is an isomorphism.

   To prove injectivity, let $[(s,U)] \in \mathcal{O}_\mathfrak{p}$ such that $s(\mathfrak{p}) = 0 \in A_\mathfrak{p}$. By definition of $\mathcal{O}(U)$, after shrinking U we may assume that there exist $f, g \in A$ such that for all $\mathfrak{q} \in U$, we have $g \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{f}{g} \in A_\mathfrak{q}$. In particular, $\frac{f}{g} = 0 \in A_\mathfrak{p}$ implies $fh = 0$ for some $h \in A \setminus \mathfrak{p}$. By intersecting U with the open neighbourhood $U_h = \operatorname{Spec} A \setminus V(h)$ of $\mathfrak{p}$, we may further assume that $h \in A \setminus \mathfrak{q}$ for all $\mathfrak{q} \in U$. But then $s(\mathfrak{q}) = \frac{f}{g} = 0 \in A_\mathfrak{q}$ for all $\mathfrak{q} \in U$, i. e. $[(s,U)] = 0 \in \mathcal{O}_\mathfrak{p}$.

   To prove surjectivity, let $\frac{f}{g} \in A_\mathfrak{p}$ with $f, g \in A$ and $g \notin \mathfrak{p}$. We may then define $s \in \mathcal{O}(U_g)$ via $s(\mathfrak{q}) = \frac{f}{g} \in A_\mathfrak{q}$ for all $\mathfrak{q} \in U_g$, which is even globally a fraction. Now, $[(s,U_g)] \in \mathcal{O}_\mathfrak{p}$ maps to $s(\mathfrak{p}) = \frac{f}{g} \in A_\mathfrak{p}$. □

   b) Show that for any $f \in A$, $\mathcal{O}(U_f) = A_f$.

   Proof. Consider the ring homomorphism

$$\begin{aligned} A_f &\to \mathcal{O}(U_f) \\ \tfrac{g}{f^n} &\mapsto \left(\mathfrak{p} \mapsto \tfrac{g}{f^n} \in A_\mathfrak{p}\right) \ . \end{aligned}$$

   This is well-defined because $f^n \notin \mathfrak{p}$ for all $\mathfrak{p} \in U_f$. We prove that this map is an isomorphism.

   To prove injectivity, let $\frac{g}{f^n} \in A_f$ with $g \in A$ and $n \in \mathbb{N}$ such that $\frac{g}{f^n} = 0 \in A_\mathfrak{p}$ for all $\mathfrak{p} \in U_f$. Hence, for all $\mathfrak{p} \in U_f$ there exists some $h_\mathfrak{p} \in A \setminus \mathfrak{p}$ such that $gh_\mathfrak{p} = 0$. This yields an open cover $U_f \subset \bigcup_{\mathfrak{p} \in U_f} U_{h_\mathfrak{p}}$. In other words, $V(f) \supset V(I)$ where $I = \sum_{\mathfrak{p} \in U_f} h_\mathfrak{p} A$. In particular,

$$f \in \bigcap_{\mathfrak{q} \ni f} \mathfrak{q} \subset \bigcap_{\mathfrak{q} \supset I} \mathfrak{q} = \sqrt{I} \ ,$$

i. e. $f^m = h_{\mathfrak{p}_1} a_1 + \cdots + h_{\mathfrak{p}_k} a_k$ for some $m \in \mathbb{N}$, $\mathfrak{p}_1, \ldots, \mathfrak{p}_k \in U_f$, and $a_1, \ldots, a_k \in A$. But this implies

$$gf^m = gh_{\mathfrak{p}_1} a_1 + \cdots + gh_{\mathfrak{p}_k} a_k = 0 ,$$

so $\frac{g}{f^n} = 0 \in A_f$, as wanted.

To prove surjectivity, let $s \in \mathcal{O}(U_f)$ be given. By definition of $\mathcal{O}(U_f)$, for all $\mathfrak{p} \in U_f$ there exists an open neighbourhood $V_\mathfrak{p} \subset U_f$ of $\mathfrak{p}$ and $g_\mathfrak{p}, h_\mathfrak{p} \in A$ such that for all $\mathfrak{q} \in V_\mathfrak{p}$, we have $h_\mathfrak{p} \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{g_\mathfrak{p}}{h_\mathfrak{p}} \in A_\mathfrak{q}$. Since the standard open sets U_e for $e \in A$ form a basis of the Zariski topology on $\operatorname{Spec} A$, by shrinking $V_\mathfrak{p}$ we may assume that $V_\mathfrak{p} = U_{e_\mathfrak{p}}$ for some $e_\mathfrak{p} \in A \backslash \mathfrak{p}$. By replacing $h_\mathfrak{p}$ and $e_\mathfrak{p}$ by their product (and $g_\mathfrak{p}$ by $g_\mathfrak{p} e_\mathfrak{p}$), we may further assume that $e_\mathfrak{p} = h_\mathfrak{p}$. As in the proof of injectivity, the open cover $U_f \subset \bigcup_{\mathfrak{p} \in U_f} U_{h_\mathfrak{p}}$ gives rise to an equation $f^m = h_{\mathfrak{p}_1} a_1 + \cdots + h_{\mathfrak{p}_k} a_k$ for suitable $m \in \mathbb{N}$, $\mathfrak{p}_1, \ldots, \mathfrak{p}_k \in U_f$, and $a_1, \ldots, a_k \in A$. For simplicity, we write g_i and h_i instead of $g_{\mathfrak{p}_i}$ and $h_{\mathfrak{p}_i}$ in the following. Let $i, j \in \{1, \ldots, k\}$. Since for all $\mathfrak{q} \in U_{h_i} \cap U_{h_j} = U_{h_i h_j}$ we have $\frac{g_i}{h_i} = s(\mathfrak{q}) = \frac{g_j}{h_j}$ in $A_\mathfrak{q}$, the already proven injectivity of the map $A_{h_i h_j} \to \mathcal{O}(U_{h_i h_j})$ yields $\frac{g_i}{h_i} = \frac{g_j}{h_j}$ in $A_{h_i h_j}$, which means that $(h_i h_j)^N (g_i h_j - g_j h_i) = 0$ for sufficiently large $N \in \mathbb{N}$. Since there are only finitely many pairs (i, j), we can choose $N \in \mathbb{N}$ independent of i and j. Let $g_i' = h_i^N g_i$ and $h_i' = h_i^{N+1}$ for $i \in \{1, \ldots, k\}$. Then we have $s(\mathfrak{q}) = \frac{g_i}{h_i} = \frac{g_i'}{h_i'} \in A_\mathfrak{q}$ for all $\mathfrak{q} \in U_{h_i} = U_{h_i'}$ and $g_i' h_j' = g_j' h_i'$ for all $i, j \in \{1, \ldots, k\}$. The open cover $U_f \subset U_{h_1'} \cup \cdots \cup U_{h_k'}$ shows that $f^{m'} = h_1' a_1' + \cdots + h_k' a_k'$ for some $m' \in \mathbb{N}$ and certain $a_1', \ldots, a_k' \in A$. We now replace m, g_i, h_i, a_i by m', g_i', h_i', a_i'. We claim that the element

$$b = \frac{g_1 a_1 + \cdots + g_k a_k}{f^m} \in A_f$$

maps to s. Let $\mathfrak{q} \in U_f$, so $\mathfrak{q} \in U_{h_i}$ for at least one $i \in \{1, \ldots, k\}$. We now have

$$b = \frac{(g_1 a_1 + \cdots + g_k a_k) h_i}{f^m h_i} = \frac{(h_1 a_1 + \cdots + h_k a_k) g_i}{f^m h_i} = \frac{g_i}{h_i} = s(\mathfrak{q})$$

as desired. □

2. Recall that a ring A is a local ring if it has a unique maximal ideal $\mathfrak{m}_A$. A homomorphism between local rings $\varphi \colon A \to B$ is called a local homomorphism if $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B$.

   a) Show that a ring A is a local ring if and only if $\operatorname{Spec} A$ has a unique closed point.

      Proof. We know that the closed points of $\operatorname{Spec} A$ are precisely the maximal ideals of A. Therefore, A has a unique maximal ideal if and only if $\operatorname{Spec} A$ has a unique closed point. □

b) Show that a homomorphism $\varphi\colon A \to B$ between local rings is local if and only if the induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ maps the closed point of $\operatorname{Spec} B$ to the closed point of $\operatorname{Spec} A$.

Proof. We need to show the equivalence $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B \iff \varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$. The direction "$\Longleftarrow$" is clear. Conversely, if $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B$, we have $\mathfrak{m}_A \subset \varphi^{-1}(\mathfrak{m}_B)$. But $\mathfrak{m}_A \subset A$ is a maximal ideal and $\varphi^{-1}(\mathfrak{m}_B) \subset A$ is a prime ideal, so $\mathfrak{m}_A = \varphi^{-1}(\mathfrak{m}_B)$. $\square$

c) Give an example of a homomorphism $\varphi\colon A \to B$ between local rings which is not local. Describe the induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ in your example.

Proof. Let k be a field. Let $A = k[x]_{(x)}$ and $B = k(x)$. Then A is a local ring (because it is a localization at a prime ideal) with maximal ideal $\mathfrak{m}_A = xA$ and B is a local ring (because it is a field) with maximal ideal $\mathfrak{m}_B = (0)$. But for the canonical ring homomorphism $\varphi\colon A \to B$, we have $\varphi(\mathfrak{m}_A) = B \not\subset \mathfrak{m}_B$ because $x \in B$ is a unit, so φ is not local. The induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ corresponds to the embedding of the generic point into the irreducible one-dimensional scheme $\operatorname{Spec} A$. $\square$

d) Give examples of local rings A such that $\operatorname{Spec} A$ has exactly

i. one point;

Proof. Take $A = k$ to be a field. $\square$

ii. two points;

Proof. Take A to be a discrete valuation ring (see next exercise). $\square$

iii. infinitely many points.

Proof. Take any local ring of dimension at least two, for example $A = k[x, y]_{(x,y)}$. $\square$

3. Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$, residue field $k = R/\mathfrak{m}$ and fraction field $K = \operatorname{Frac} R$.

a) Describe the topological space $\operatorname{Spec} R$.

Proof. Since R is a one-dimensional integral domain, the only other prime ideal apart from the maximal ideal $\mathfrak{m}$ is the zero ideal, hence $\operatorname{Spec} R = \{(0), \mathfrak{m}\}$. We know that the generic point $(0) \in \operatorname{Spec} R$ is dense and the point $\mathfrak{m} \in \operatorname{Spec} R$ corresponding to the maximal ideal is closed. Hence, the open subsets of $\operatorname{Spec} R$ are precisely $\emptyset$, $\{(0)\}$ and $\operatorname{Spec} R$. $\square$

b) Compute the structure sheaf $\mathcal{O}$ on any open subset of $\operatorname{Spec} R$.

Proof. We know that $\mathcal{O}(\emptyset) = 0$ and $\mathcal{O}(\operatorname{Spec} R) = R$. And by the first exercise, $\mathcal{O}(\{(0)\}) = \mathcal{O}_{(0)} = A_{(0)} = K$. □

c) Compute all stalks of the structure sheaf $\mathcal{O}$ on $\operatorname{Spec} R$.

Proof. By the first exercise, $\mathcal{O}_{(0)} = A_{(0)} = K$ and $\mathcal{O}_{\mathfrak{m}} = R_{\mathfrak{m}} = R$. (The corresponding residue fields would be K and k.) □

4. a) Give an example of a ring A and a non-zero element $f \in \mathcal{O}(\operatorname{Spec} A)$ such that for all $\mathfrak{p} \in \operatorname{Spec} A$, the value of f in the residue field $\kappa(\mathfrak{p})$ vanishes.

   b) Can you give an example as above in the case where $A = k[X]$ for an affine variety X over an algebraically closed field k? If not, try to pin down the ring theoretic property that $k[X]$ has which excludes a phenomenon as above.

Proof. Note that an element $f \in \mathcal{O}(\operatorname{Spec} A) = A$ satisfies $f(\mathfrak{p}) = 0 \in \kappa(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} A$ if and only if $f \in \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} = \sqrt{(0)}$, i. e. f is nilpotent.

Therefore, for (a) we can take any non-reduced ring A and any non-zero nilpotent element $f \in A$. For example, if $A = k[x]/(x^2)$ and $f = x$, we have $f((x)) = 0 \in k = \kappa((x))$ at the only point $(x) \in \operatorname{Spec} A$.

Conversely, no such examples exist if A is reduced, in particular, if $A = k[X]$. □