Wintersemester 2018/19

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Algebraic Geometry

Sheet 8

Unless specified otherwise, we will always work over an algebraically closed field k.

Exercise 1. (4 points) *A family of cubic curves*
For $t \in k$, consider the cubic curve

$$X_t := V_{\mathbb{P}^2}(x_0^3 + x_1^3 + x_2^3 + t(x_0 + x_1 + x_2)^3) \subset \mathbb{P}^2.$$

For which values of t is X_t singular and what are the corresponding singular points?

Exercise 2. (4 points) *Singular points on cubic curves*
Let $X \subset \mathbb{P}^2$ be a hypersurface of degree 3, i.e. $X = V_{\mathbb{P}^2}(F)$ where $F \in k[x_0, x_1, x_2]$ is a homogeneous polynomial of degree three.

(a) Show that if X has two singular points $x, y \in X$ with $x \neq y$, then the line joining x and y is an irreducible component of X.

(b) Deduce that X has at most 3 singular points. Show further that if X has exactly three singular points, then it is the union of three lines.

(c) Show that X has at most one singular point if it is irreducible.

Exercise 3. (4 points) *Smooth hypersurfaces.*
Let d, n be positive integers. Find an example of an irreducible homogeneous polynomial $F \in k[x_0, \dots, x_{n+1}]$ of degree d such that $X = V_{\mathbb{P}^{n+1}}(F) \subset \mathbb{P}^{n+1}$ is smooth.
Hint: For simplicity, you may assume that the characteristic of k does not divide d, but the desired examples exist without that assumption.

Exercise 4. (4 points) *The tangent space of a union of two components.*
Let $X = X_1 \cup X_2$ be the union of two affine varieties $X_1, X_2 \subset \mathbb{A}^n$ with $X_i \not\subset X_j$ for $i \neq j$ and let $x \in X_1 \cap X_2$ be a point in the intersection of X_1 and X_2.

(a) Show that x is a singular point of X.

Hint: You can use that the local ring $\mathcal{O}_{X,x}$ at a smooth point $x \in X$ is regular, that is, its maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is generated by $\dim_x(X)$ elements. This statement has accidentally been skipped in the lecture and it will be catched up next week in class.

(b) Show that $T_{X_1,x} + T_{X_2,x} \subset T_{X,x}$. Is it always true that equality holds here?

Hand in: before noon on Monday, December 10th in the appropriate box on the 1st floor.