Wintersemester 2018/19

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Algebraic Geometry

Sheet 13

Unless specified otherwise, we will always work over an algebraically closed field k.

Exercise 1. (4 points) *The Picard group.*
Let X be a quasi-projective variety and let $\mathcal{L}$ and $\mathcal{L}'$ be line bundles on X, i.e. a locally free $\mathcal{O}_X$-modules of rank one on X.

(a) Show that $\mathcal{L} \cong \mathcal{O}_X$ if and only if there is a global section $s \in H^0(X, \mathcal{L})$ such that for all $x \in X$, the image

$$s(x) \in \mathcal{L}(x) := \mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} (\mathcal{O}_{X,x}/\mathfrak{m}_x)$$

of s in the fibre $\mathcal{L}(x)$ of $\mathcal{L}$ at x is nonzero.

(b) Show that the tensor product $\mathcal{L} \otimes \mathcal{L}'$, defined as sheafification of the presheaf $U \mapsto \mathcal{L}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{L}'(U)$, is locally free of rank one, hence a line bundle on X.

(c) Show that the dual

$$\mathcal{L}^\vee := \mathcal{H}om(\mathcal{L}, \mathcal{O}_X),$$

defined as sheafification of the presheaf $U \mapsto \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{L}(U), \mathcal{O}_X(U))$ is locally free of rank one, hence a line bundle on X.

(d) Show that

$$\mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{O}_X.$$

(e) Show that the tensor product of line bundles makes the set of isomorphism classes of line bundles on X into an abelian group $\mathrm{Pic}(X)$ with neutral element the trivial line bundle $\mathcal{O}_X$.

Remark. The group $\mathrm{Pic}(X)$ is called Picard group of X.

Exercise 2. (4 points) *Cartier divisors.*
Let X be a normal quasi-projective algebraic variety. We say that a divisor $D \in \mathrm{Div}(X)$ is Cartier if the corresponding $\mathcal{O}_X$-module $\mathcal{O}_X(D)$ is a locally free $\mathcal{O}_X$-module.

(a) Let $D \in \mathrm{Div}(X)$ be Cartier. Show that $\mathcal{O}_X(D)$ is locally free of rank one.

(b) Show that a divisor $D \in \mathrm{Div}(X)$ is Cartier if and only it is locally given by the zeros and poles of a single rational function. That is, if and only if for each $x \in X$, there is an affine open neighbourhood $x \in U \subset X$ of x and a nonzero rational function $\varphi \in k(X)^\times$ such that $\mathrm{Div}(\varphi)|_U = D|_U$.

Exercise 3. (4 points) *A divisor which is not Cartier.*
Consider the example of the quadric cone

$$X := V_{\mathbb{A}^3}(x_1x_2 - x_3^2) \subset \mathbb{A}^3.$$

(a) Show that the singular locus of X is zero-dimensional. Use this to deduce that X is normal.

(**Hint:** You may use without proof that a hypersurface $X \subset \mathbb{A}^n$ is normal if and only if the singular locus X^{sing} has codimension at least two in X, i.e. $\dim(X) \geq \dim(X^{\text{sing}}) + 2$.)

(b) Let $D := V_X(x_1, x_3)$. Show that D is a prime divisor on X, i.e. D is irreducible of codimension one on X.

(c) Show that D from above is not a Cartier divisor, i.e. $\mathcal{O}_X(D)$ is not locally free.

Exercise 4. (4 points) *The Fermat elliptic curve*
Let $X := V_{\mathbb{P}^2}(x_0^3 + x_1^3 + x_2^3)$. Show that X carries a regular differential form $\omega \in H^0(X, \omega_X)$ such that for all $x \in X$, the linear form $\omega(x) \in T^*_{X,x}$ is nonzero. Deduce that $\omega_X \cong \mathcal{O}_X$ is trivial and so $g(X) = 1$.

Hand in: before noon on Monday, February 4th in the appropriate box on the 1st floor.