Wintersemester 2018/19

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Algebraic Geometry

Sheet 13

Unless specified otherwise, we will always work over an algebraically closed field k.

Exercise 1. (4 points) *Locally free $\mathcal{O}_X$-modules are coherent.*

(a) Let X be an affine algebraic variety with ring of regular functions $R = k[X]$. Show that there is a natural isomorphism of $\mathcal{O}_X$-modules

$$\widetilde{R} \xrightarrow{\sim} \mathcal{O}_X.$$

(b) Let X be a quasi-projective variety and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. Show that $\mathcal{E}$ is coherent (hence also quasi-coherent).

Exercise 2. (4 points) *Restriction of quasi-coherent sheaves to standard open subsets.*

(a) Let $\mathcal{F}$ be a sheaf of abelian groups on a toppological space X. Let $i : U \hookrightarrow X$ be the inclusion of an open subset $U \subset X$. Recall that the restriction of $\mathcal{F}$ to U is defined via $\mathcal{F}|_U := i^{-1}\mathcal{F}$. Show that

$$\mathcal{F}|_U(V) = \mathcal{F}(V)$$

for all open subsets $V \subset U$.

(b) Let X be an affine variety, let $R = k[X]$ be its ring of regular functions and let M be an R-module. Let $f \in R$ and let $U_f := X \setminus V_X(f)$ be the corresponding standard open subset. Prove that

$$\widetilde{M}|_{U_f} \cong \widetilde{M_f}.$$

That is, the restriction of the $\mathcal{O}_X$-module $\widetilde{M}$ to U_f is naturally isomorphic to the $\mathcal{O}_{U_f}$-module which corresponds to the $k[U_f] = R_f$-module M_f.

Remark: You should prove this directly, without using Exercise 3 below (whose solution most likely will use the result you prove here).

Exercise 3. (4 points) *Right exactness of global section functor for quasi-coherent sheaves on affine varieties.*
Let X be an affine variety with ring of regular functions $R = k[X]$ and let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module on X.

(a) Show that there is a finite covering $X = \bigcup_{i=}^{n} U_i$ of X by standard open subsets $U_i = U_{g_i}$, such that $\mathcal{M}|_{U_i} \cong \widetilde{M_i}$ for some R_{g_i}-module M_i.

(b) Let $s \in \mathcal{M}(X)$ be a global section and let $f \in R$ be a nonzero regular function on X. Show that $s|_{U_f} = 0$ implies that $f^m s = 0$ for some $m \geq 0$.

(c) Let $f \in R$ be a nonzero regular function on X and let $s \in \mathcal{M}(U_f)$ be a section of $\mathcal{M}$ over U_f. Show that for some $m \geq 0$, the section $f^m s \in \mathcal{M}(U_f)$ extends to X, i.e. it lies in the image of the restriction map $\mathcal{M}(X) \to \mathcal{M}(U_f)$.

(d) Use the above items to show that there is a natural isomorphism

$$\phi : \widetilde{\mathcal{M}(X)} \xrightarrow{\sim} \mathcal{M}.$$

(e) Conclude from part (d) that for any short exact sequence $0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$ of quasi-coherent $\mathcal{O}_X$-modules on X, the corresponding sequence on global sections

$$0 \to \mathcal{M}_1(X) \to \mathcal{M}_2(X) \to \mathcal{M}_3(X) \to 0$$

is again exact.

Exercise 4. (4 points) *Differential forms*

(a) Let R be a finitely generated k-algebra. Show that the module of Kähler differentials $\Omega^1_{R/k}$ is a finitely generated R-module.

(b) Let $R = k[t_1, \ldots, t_n]$ be the polynomial ring over k in n variables. Show that the symbols $dt_1, \ldots, dt_n$ form a basis of the module of Kähler differentials $\Omega^1_{R/k}$. In particular, $\Omega^1_{R/k} \cong R^{\oplus n}$ is free of rank n.

(c) Let $R = k[X]$ be the coordinate ring of the affine variety $X = V(t_1^2 - t_2^3) \subset \mathbb{A}^2$. Show that the module of Kähler differentials $\Omega^1_{R/k}$ is not a free R-module.

Hand in: before noon on Monday, January 28th in the appropriate box on the 1st floor.