Wintersemester 2018/19

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Algebraic Geometry

Sheet 12

Unless specified otherwise, we will always work over an algebraically closed field k.

Exercise 1. (4 points) *The direct image sheaf*
Let $f : X \to Y$ be a continuous map of topological spaces and let $\mathcal{F}$ be a sheaf of abelian groups on X. Consider the presheaf $f_*\mathcal{F}$ on Y which on an open subset $V \subset Y$ is given by

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

and whose restriction maps are induced by those of $\mathcal{F}$.

(a) Show that $f_*\mathcal{F}$ is a sheaf.

(b) Compute the stalks of $f_*\mathcal{F}$ in the following examples:

(i) f is constant, i.e. there is a point $y_0 \in Y$ with $f(x) = y_0$ for all $x \in X$;

(ii) $X = Y = \mathbb{R}$ with the euclidean topology and $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$.

Exercise 2. (4 points) *The inverse image sheaf.*
Let $f : X \to Y$ be a continuous map of topological spaces and let $\mathcal{G}$ be a sheaf of abelian groups on Y. Consider the sheaf $f^{-1}\mathcal{G}$ on X which is given as sheafification of the presheaf which on an open subset $U \subset X$ is given by the direct limit

$$\lim_{V \supset f(U)} \mathcal{G}(V),$$

where V runs through all open subsets of Y which contain $f(U)$ and whose restriction maps are induced by those of $\mathcal{G}$.

(a) Show that $f^{-1}\mathcal{G}_x \cong \mathcal{G}_{f(x)}$ for all $x \in X$.

(b) Show that for any abelian group G, we have $f^{-1}\underline{G}_Y \cong \underline{G}_X$

(c) Show that f^{-1} is a functor. That is, for any morphism of sheaves $\varphi : \mathcal{G} \to \mathcal{G}'$ there is a natural morphism $f^{-1}(\varphi) : f^{-1}\mathcal{G} \to f^{-1}\mathcal{G}'$.

(**Hint:** You may use that sheafification is functorial, i.e. any morphism of presheaves $\psi : \mathcal{F}_1 \to \mathcal{F}_2$ induces a unique morphism of sheaves $\psi^+ : \mathcal{F}_1^+ \to \mathcal{F}_2^+$.)

(d) Show that the functor f^{-1} is exact. That is, for any exact sequence of sheaves $\mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3$ on Y the induced sequence $f^{-1}\mathcal{G}_1 \to f^{-1}\mathcal{G}_2 \to f^{-1}\mathcal{G}_3$ on X is exact.

Definition. Let X be a quasi-projective algebraic variety. A (algebraic) vector bundle of rank r on X is a quasi-projective variety E together with a regular map $\pi : E \to X$ with the following properties:

(1) there is an open covering $X = \bigcup_{i \in I} U_i$ of X and there are isomorphisms

$$\varphi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times k^r$$

that are compatible with the two natural projections to U_i, i.e. $\mathrm{pr}_1 \circ \varphi_i = \pi|_{\pi^{-1}(U_i)}$.

(2) for all $i, j \in I$ the composition

$$\varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times k^r \longrightarrow (U_i \cap U_j) \times k^r$$

is of the form

$$\varphi_j \circ \varphi_i^{-1}(x, v) = (x, \varphi_{ij}(x) \cdot v),$$

where $x \in U_{ij}$, $v \in k^r$, and where $\varphi_{ij}(x) \in \mathrm{GL}(r, k)$ is an invertible $r \times r$ matrix over the field k whose entries are regular functions on U_{ij}.

Exercise 3. (4 points) *Algebraic vector bundles.*
Let X be a quasi-projective algebraic variety and let $\pi : E \to X$ be an algebraic vector bundle of rank r on X.

(a) Show that $\dim E = \dim X + r$.

(b) Let $E(x) = \pi^{-1}(x)$ be the fibre of π above x. Choose an index $i \in I$ with $x \in U_i$ and define the structure of a k-vector space on $E(x)$ via the isomorphism $E(x) \cong k^r$ that is induced by φ_i. Show that the vector space structure thus defined does not depend on i.

(c) Let $s_0 : X \to E$ be the map which sends $x \in X$ to the origin in the vector space $E(x)$ (with vector space structure defined in (b) above). Show that s_0 is a regular map with $\mathrm{id}_X = \pi \circ s_0$.

Exercise 4. (4 points) *Sheaf of regular sections of a vector bundle.*
Let X be a quasi-projective algebraic variety and let $\pi : E \to X$ be an algebraic vector bundle of rank r on X. Let $\mathcal{E}$ be the presheaf on X given by the regular sections of π, i.e. for any nonempty open subset $U \subset X$ $\mathcal{E}(U)$ is given as a set by all regular maps $s : U \to E$ with $\pi \circ s = \mathrm{id}_U$ and where the group law is induced by pointwise addition (which uses that the fibre $E(x)$ above $x \in X$ is a k-vector space).

(a) Show that $\mathcal{E}$ is a sheaf.

(b) Show that $\mathcal{E}$ carries naturally the structure of an $\mathcal{O}_X$-module.

(c) Show that $\mathcal{E}$ is in fact a locally free $\mathcal{O}_X$-module.

(**Remark:** One can show conversely that any locally free $\mathcal{O}_X$-module $\mathcal{M}$ of rank r is isomorphic to the sheaf of regular sections of an algebraic vector bundle of rank r on X.)

Hand in: before noon on Monday, January 21st in the appropriate box on the 1st floor.