Wintersemester 2018/19

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Algebraic Geometry

Sheet 11

Unless specified otherwise, we will always work over an algebraically closed field k.

Exercise 1. (4 points) *The sheaf of rational functions.*
Let X be a quasi-projective variety. Consider the presheaf $\mathcal{K}$ on X whose sections on a non-empty open subset $U \subset X$ are given by

$$\mathcal{K}(U) = \{f : U \dashrightarrow k \mid f \text{ is a rational function}\}.$$

(a) Show that $\mathcal{K}$ is a sheaf.

(b) Show that $\mathcal{K}$ is isomorphic to the constant sheaf $\underline{k(X)}_X$ on X with values in the function field $k(X)$.

Exercise 2. (4 points) *The sheaf associated to a divisor.*
Let X be a normal quasi-projective variety and let $D \in \operatorname{Div}(X)$ be a divisor on X. Consider the presheaf $\mathcal{O}_X(D)$ on X which on a nonempty open subset $U \subset X$ is given by

$$\mathcal{O}_X(D)(U) := \{f \in k(X) \mid \operatorname{Div}(f)|_U + D|_U \geq 0\}.$$

In other words, the sections of $\mathcal{O}_X(D)$ over a nonempty open subset $U \subset X$ are given by all rational functions f on X such that the divisor $\operatorname{Div}(f) + D$ is effective when restricted to U, i.e. $\operatorname{Div}(f) + D = D' + D''$ for an effective divisor D' and a divisor D'' which is supported on $X \setminus U$.

(a) Show that $\mathcal{O}_X(D)$ is a sheaf of abelian groups. Show further that $\mathcal{O}_X(D)$ is an $\mathcal{O}_X$-module, i.e. for any nonempty open subset $U \subset X$, $\mathcal{O}_X(D)(U)$ is a module over the ring $\mathcal{O}_X(U)$ and this module structure is compatible with the restriction maps on both sides.

(b) Let $D = 0$ be the trivial divisor. Show that $\mathcal{O}_X(D) \cong \mathcal{O}_X$.

(c) Let D_1 and D_2 be two divisors on X with $D_1 \sim D_2$, i.e. D_1 and D_2 have the same class in $\operatorname{Cl}(X)$. Show that $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$. In particular, $\mathcal{O}_X(D) \cong \mathcal{O}_X$ if $D \sim 0$.

(d) Let $D \in \operatorname{Div}(X)$ be a divisor such that there is an isomorphism $\varphi : \mathcal{O}_X(D) \xrightarrow{\sim} \mathcal{O}_X$ of sheaves which is compatible with the natural $\mathcal{O}_X$-module structures on both sides. Prove that $D \sim 0$.

Exercise 3. (4 points) *Surjective morphisms of sheaves.*

(a) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups on a topoplogical space X. Show that φ is surjective if and only if for all open subsets $U \subset X$ and for every $s \in \mathcal{G}(U)$ there is a covering $U = \bigcup U_i$ of U by open subsets U_i and sections $t_i \in \mathcal{F}(U_i)$ with $\varphi_{U_i}(t_i) = s|_{U_i}$ for all i.

(b) Give an example of a topological space X and a surjective morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ of abelian groups on X such that for some open subset $U \subset X$,

$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

is not surjective.

Exercise 4. (4 points) *Stalks.*

(a) Let G be an abelian group and let X be a topological space. Compute the stalks of the following sheaves on X

(i) the constant sheaf $\underline{G}_X$ with values in G on X;

(ii) the skyscraper sheaf $\underline{G}_{\{x\}}$ concentrated on a point $x \in X$.

(b) Let $\pi : E \to X$ be a real topological vector bundle. That is, π is a continuous map of topological spaces and there is an open covering $X = \bigcup_{i \in I} U_i$ such that there are homeomorphisms $\varphi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{R}^r$ with $\pi|_{\pi^{-1}(U_i)} = \mathrm{pr}_1 \circ \varphi_i$, and such that additionally for any i, j the transition map

$$\varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^r \to (U_i \cap U_j) \times \mathbb{R}^r$$

is of the form $(x, v) \mapsto (x, \varphi_{ij}(x) \cdot v)$, where $\varphi_{ij}(x) \in \mathrm{GL}_r(\mathbb{R})$ depends continuously on x. (This implies in particular that the fibre $\pi^{-1}(x)$ has a well-defined structure of a real vector space.)

Consider the presheaf $\mathcal{E}$ on X whose sections over $U \subset X$ are all continuous maps $s : U \to E$ with $\pi \circ s = \mathrm{id}_U$.

(i) Show that the stalks of $\mathcal{E}$ and the stalks of the sheaf $\mathcal{C}^0_{X,\mathbb{R}^k}$ are isomorphic.

(ii) In the previous exercise, specialize to the case where $X = S^1$ is the circle and E is the real rank one vector bundle which corresponds to the Moebius strip. Show that there is no isomorphism of sheaves $\varphi : \mathcal{E} \to \mathcal{C}^0_{S^1,\mathbb{R}}$ which is compatible with the natural $\mathcal{C}^0_{S^1,\mathbb{R}}$-module structure that we have on both sides, i.e. $\varphi_U(fs) = f\varphi_U(s)$ for any $f \in \mathcal{C}^0_{S^1,\mathbb{R}}(U)$, $s \in \mathcal{E}(U)$ and any $U \subset X$ open.

Hint: You may use without proof that $E \setminus s_0(S^1)$ is connected, where $s_0 : S^1 \to E$ is the zero section, e.g. $s_0 = 0 \in \mathcal{E}(S^1)$.

Hand in: before noon on Monday, January 14th in the appropriate box on the 1st floor.