Wintersemester 2018/19

Prof. Dr. Stefan Schreieder
Dr. Feng Hao

Algebraic Geometry

Sheet 10

Unless specified otherwise, we will always work over an algebraically closed field k.

Exercise 1. (4 points) *Rational maps on normal varieties*
Let X be a normal quasi-projective variety and let $Y \subset \mathbb{P}^m$ be a projective variety. Let $f : X \dashrightarrow Y$ be a rational map. Show that f is regular in codimensino one, that is,

$$\dim(X \setminus \mathrm{dom}(f)) \leq \dim(X) - 2.$$

Hint: Reduce to the case where X is affine and $Y = \mathbb{P}^m$. Assume for a contradiction that $Z \subset X$ is an irreducible closed subset of codimension one such that $Z \cap \mathrm{dom}(f) = \emptyset$. Deduce a contradiction from the fact that $\mathcal{O}_{X,Z}$ is normal of dimension one, hence a discrete valuation ring.

Exercise 2. (4 points) *Birational smooth projective curves are isomorphic*
Let X and Y be smooth projective curves, i.e. smooth projective varieties of dimension one. Show that X and Y are isomorphic if and only if they are birationally equivalent.

Hint: Use Exercise 1.

Exercise 3. (4 points) *Class group of $\mathbb{P}^n$.*
Let $n \geq 1$ be a positive integer. Show that $\mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z}[H]$, where $H \subset \mathbb{P}^n$ is a hyperplane, i.e. a subvariety given by the vanishing of a homogeneous polynomial of degree one.

Exercise 4. (4 points) *The class group of smooth projective curves.*
Let X be a smooth projective curve, i.e. a smooth projective variety of dimension one. Let $\mathrm{Cl}^0(X)$ be the kernel of the degree map $\deg : \mathrm{Cl}(X) \to \mathbb{Z}$. For a fixed base point $x_0 \in X$, consider the map

$$\Phi : X \longrightarrow \mathrm{Cl}^0(X), \quad x \mapsto x - x_0.$$

Show that the above map is injective, unless X is isomorphic to $\mathbb{P}^1$. In particular, $\mathrm{Cl}(X) \cong \mathbb{Z}$ holds if and only if $X \cong \mathbb{P}^1$.

Hint: If Φ is not injective, then there are two different points $x, y \in X$ with $x \sim y$, i.e. there is a rational function φ with $\mathrm{Div}(\varphi) = x - y$. Show that φ corresponds to a rational map $X \dashrightarrow \mathbb{P}^1$, which is a morphism by Exercise 1. Use the Fact about degrees of maps between smooth projective curves from the lecture to conclude that φ is an isomorphism.

Exercise 5. (10 extra points) *A collection of small exercises to repeat what you have learned so far.*

(a) Let $f : X \to Y$ be a regular map between quasi-projective varieties. Is it true that f is an isomorphism if it is bijective?

(b) Consider the subrings $R := k[t^3, t^7]$ and $S := k[t^3, t^4]$ of the polynomial ring $k[t]$. Let $\varphi : R \to S$ be the inclusion. Find affine varieties X and Y with $k[X] = S$ and $k[Y] = R$ and a regular map $f : X \to Y$ such that $f^* : k[Y] \to k[X]$ coincides with φ.

(c) Let $f : X \to Y$ be a regular map between quasi-projective varieties. Suppose that there is a point $y \in Y$ such that $f^{-1}(y) = \{x\}$ consists of a single point. Show that $\dim Y \geq \dim X$. Show further that $\dim Y = \dim X$ if f is dominant. Is it true that in this situation f is necessarily birational, if it is dominant?

(d) Give an example of a birational map between projective varieties which is not an isomorphism.

(e) Give an example of a projective variety which has exactly one singular point. Can you similarly give an example with n singular points, where n is any given natural number?

(f) Let X be an affine variety. Assume that X is isomorphic to a projective variety. Show that X is a point.

(g) Let X be a quasi-projective variety. Show that

$$\dim X = \min\{\dim(T_{X,x}) \mid x \in X)\}.$$

(h) Compute the normalization of the affine variety $Y := V(x_1^2 - x_2^5) \subset \mathbb{A}^2$.

(i) Let $X = V(F) \subset \mathbb{P}^n$ be the projective set defined by a homogeneous polynomial F. Show that X is irreducible, if it is smooth.

(j) Assume that the ground field k has characteristic different from 2. Determine whether the following projective algebraic sets X are smooth, and compute its singular points if X is singular:

(1) $X = V_{\mathbb{P}^3}(x_0^2 - x_0x_2 - x_1x_3,\ x_1x_2 - x_0x_3 - x_2x_3)$;

(2) $X = V_{\mathbb{P}^4}(x_0x_1 - x_2^2 - x_3^2,\ x_0x_1 + x_2x_3 + x_4^2)$.

Hand in: before noon on Monday, January 7th in the appropriate box on the 1st floor.