Wintersemester 2018/19

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Algebraic Geometry

Sheet 1

Unless specified otherwise, we will always work over an algebraically closed field k.

Exercise 1. (4 points) *Basic properties of ideals of sets.*

(a) For an affine algebraic set $X = V(I)$, prove that $I \subset I(X)$.

(b) Let $X', X'' \subset \mathbb{A}^n_k$ be affine algebraic sets.

(i) Show that if $X' \subset X''$ then $I(X'') \subset I(X')$.

(ii) Show further that $I(X' \cup X'') = I(X') \cap I(X)$.

(iii) Show that it may happen that $I(X' \cap X'') \neq I(X') + I(X'')$.

Exercise 2. (4 points) *Intersecting the circle with a line.*
Let k be a field of characteristic different from 2.

(a) Let $X = V(x_1^2 + x_2^2 - 1, x_1) \subset \mathbb{A}^2_k$. Draw a picture of X. Compute $I(X)$ and decide whether it is a prime ideal or not.

(b) Let $X = V(x_1^2 + x_2^2 - 1, x_1 - 1) \subset \mathbb{A}^2_k$. Draw a picture of X. Compute $I(X)$ and decide whether it is a prime ideal or not.

Exercise 3. (4 points) *Basic properties of the Zariski toplogy.*

(a) Let X be an irreducible topological space (e.g. an affine algebraic variety). Show that any two nonempty open subsets of X have non-empty intersection. Show further that for any non-empty open subset $U \subset X$, we have $\overline{U} = X$.

(b) Identifying $\mathbb{A}^2_k$ with $\mathbb{A}^1_k \times \mathbb{A}^1_k$ in a natural way, show that the Zariski topology on $\mathbb{A}^2_k$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1_k$.

Exercise 4. (4 points) *The vanishing locus of a single polynomial.*
Let $f \in k[x_1, \ldots, x_n]$ be a nonconstant polynomial.

(a) Show that if $n \geq 1$, then $|\mathbb{A}^n_k \setminus V(f)| = \infty$;

(b) Show that $|V(f)| = \infty$ if and only if $n \geq 2$.

Hand in: before noon on Monday, October 22nd in the appropriate box on the 1st floor.