Wintersemester 2018/19

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# Algebraic Geometry

Sheet 9

Unless specified otherwise, we will always work over an algebraically closed field $k$.

**Exercise 1.** (4 points) *Singular points*
Recall that a point of a variety is singular if it is not smooth. Determine the singular points of the following projective varieties:

(a) $V(x_1^6 + x_2^6 - x_0^4 x_1 x_2) \subset \mathbb{P}^2$;

(b) $V(x_1^4 + x_2^4 - x_1^3 x_0) \subset \mathbb{P}^2$.

(c) $V(x_0^3 + x_3^3 + x_0 x_2 x_3) \subset \mathbb{P}^3$.

**Remark.** You may use the easy fact that has been shown in one of the exercise classes that if $F \in k[x_0, \ldots, x_{n+1}]$ is an irreducible homogeneous polynomial of degree $d$ and $X := V_{\mathbb{P}^{n+1}}(F) \subset \mathbb{P}^{n+1}$ is its zero set, then

$$X^{sing} = V_{\mathbb{P}^{n+1}}\left(F, \frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_{n+1}}\right) \subset \mathbb{P}^{n+1}.$$

**Exercise 2.** (4 points) *Finite maps*

(a) Let $\ell \geq 1$ be a positive integer. Show that $f : \mathbb{A}^1 \to \mathbb{A}^1$, $t \mapsto t^\ell$ is a finite map.

(b) Let $x \in \mathbb{A}^1$ be a point with $x \neq 0$ and let $X := \mathbb{A}^1 \setminus \{x\}$. Consider the regular map

$$f : X \to \mathbb{A}^1, \quad t \mapsto t^\ell.$$

Show that $f$ is surjective but not finite.

(c) Let $Z \subset X$ be a closed subvariety of a quasi-projective variety $X$ and let $f : Z \hookrightarrow X$ be the inclusion. Show that $f$ is a finite map. Conclude that not every finite map is surjective.

**Hint:** For simplicity, you may assume that the characteristic of $k$ does not divide $d$, but the desired examples exist without that assumption.

**Exercise 3.** (4 points) *Normalizations.*

(a) Let $X = \mathbb{A}^1$ and $Y = V(x_2^2 - x_1^3) \subset \mathbb{A}^2$ and assume that char $k \neq 2, 3$. Show that the regular map

$$f : X \to Y, \quad t \mapsto (t^2, t^3)$$

is the normalization of $Y$. That is, $f$ is finite and birational and $X$ is smooth. Draw a picture of $X$ and $Y$.

(b) Let $Y = V(x_2^2 - x_1^2(x_1 + 1)) \subset \mathbb{A}^2$. Find a normal variety $X$ together with a finite birational map $f : X \to Y$. Draw a picture of $X$ and $Y$.

**Exercise 4.** (4 points) *Tangent map*

Let $f : X \to Y$ be a regular map between quasi-projective algebraic varieties.

(a) Show that for any $x \in X$, $f$ induces a linear map

$$d_x f : T_{X,x} \to T_{Y,f(x)}.$$

(**Hint:** Reduce to the case where $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are affine and consider $d_x f := (d_x f_1, \ldots, d_x f_m)$, where $f = (f_1, \ldots, f_m)$ for regular functions $f_i \in k[X]$.)

(b) Let now $f$ be the map from item (a) in Exercise 3. Compute the rank of the differential $d_x f$ for all $x \in X$.

**Hand in:** before noon on Monday, December 17th in the appropriate box on the 1st floor.