Wintersemester 2018/19

Prof. Dr. Stefan Schreieder
Dr. Feng Hao

Algebraic Geometry

Sheet 7

Unless specified otherwise, we will always work over an algebraically closed field k.

Exercise 1. (4 points) *Cones*
Let $L \subset \mathbb{P}^{n+1}$ a linear subspace of dimension n. Let further $Y \subset L \cong \mathbb{P}^n$ be a projective variety and let $p \in \mathbb{P}^{n+1}$ be a point that does not lie on L. Let $X \subset \mathbb{P}^{n+1}$ be the union of all points on all lines that join p with some point on Y.
Prove that X is a projective variety of dimension $\dim X = \dim Y + 1$.
Remark: X is called the cone over Y with vertex p.

Exercise 2. (4 points) *Quadrics*
Let k be of characteristic different from two and let $q = \sum_{ij} a_{ij} x_i x_j \in k[x_0, x_1, x_2, \ldots, x_{n+1}]$ be a quadratic polynomial with $a_{ij} = a_{ji} \in k$. Denote the corresponding symmetric matrix by $A := (a_{ij}) \in k^{n \times n}$ and let

$$X := V_{\mathbb{P}^{n+1}}(q) \subset \mathbb{P}^{n+1}.$$

(a) Show that the isomorphism type of X depends only on the rank of A. In particular, X is isomorphic to $V(x_0^2 + \cdots + x_{\operatorname{rk} A}^2) \subset \mathbb{P}^{n+1}$ and so it is isomorphic to a cone over a smaller dimensional quadric if $\operatorname{rk} A < n+2$.

Hint: Perform a linear transformation and remember what you have learned in linear algebra.

(b) Let $n = 2$ and suppose that A has full rank. Show that X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ if $\operatorname{rk} A = 4$.

(c) Show that X is isomorphic to a union of two different hyperplanes in $\mathbb{P}^{n+1}$ if $\operatorname{rk} A = 2$.

(d) Show that X is isomorphic to a single hyperplane if $\operatorname{rk} A = 1$.

Exercise 3. (4 points) $\mathbb{P}^1 \times \mathbb{P}^1$ *versus* $\mathbb{P}^2$
Show that $\mathbb{P}^1 \times \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$.
Hint: Recall that we have proven in the lecture that $V_{\mathbb{P}^2}(F) \cap V_{\mathbb{P}^2}(G) \neq \emptyset$ for non-constant homogeneous polynomials $F, G \in k[x_0, x_1, x_2]$.

Exercise 4. (4 points) *Affine versus projective space*
Let $n, m \geq 1$ be positive integers. Show that the projection $p : \mathbb{A}^n \times \mathbb{A}^m \to \mathbb{A}^m$ to the second factor does not take closed sets to closed sets.

Hand in: before noon on Monday, December 3rd in the appropriate box on the 1st floor.