Wintersemester 2018/19

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Algebraic Geometry

Sheet 6

Unless specified otherwise, we will always work over an algebraically closed field k.

Exercise 1. (8 points) *The twisted cubic*
Consider the regular map

$$f : \mathbb{P}^1 \to \mathbb{P}^3, \quad [x_0 : x_1] \mapsto [x_0^3, x_0^2 x_1 : x_0 x_1^2 : x_1^3].$$

(a) Show that the image $T := f(\mathbb{P}^1)$ is closed and irreducible in $\mathbb{P}^3$ and so it is a projective variety.

Remark: T is called twisted cubic in $\mathbb{P}^3$ – the term 'cubic' stems from the fact that $T \subset \mathbb{P}^3$ has degree three in the sense that it intersects a general hyperplane $H \subset \mathbb{P}^3$, i.e. a hypersurface $H = V_{\mathbb{P}^3}(h)$ cut out by a general homogeneous polynomial h of degree one, in three points.

(b) Show that T is isomorphic to $\mathbb{P}^1$.

(c) Consider the rational map

$$\varphi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2, \quad [z_0 : z_1 : z_2 : z_3] \mapsto [z_0 + z_3 : z_1 : z_2].$$

Show that $T \cap \mathrm{dom}(\varphi) \neq \emptyset$. Show further that the composition

$$g := \varphi \circ f : \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$$

is in fact a regular map.

(d) Show that $g(\mathbb{P}^1) \subset \mathbb{P}^2$ is closed and compute all fibres of $g : \mathbb{P}^1 \to g(\mathbb{P}^1)$.

(Draw a picture of what you find!)

Exercise 2. (4 points) *Blow-up*
Let $n \geq 1$ and consider the rational map

$$\varphi : \mathbb{A}^{n+1} \dashrightarrow \mathbb{P}^n, \quad (x_0, \ldots, x_n) \mapsto [x_0 : \cdots : x_n].$$

Let $Bl_0\mathbb{A}^{n+1} \subset \mathbb{A}^{n+1} \times \mathbb{P}^n$ be the closure of

$$\{(x, \varphi(x)) \in \mathbb{A}^{n+1} \times \mathbb{P}^n \mid x \in \mathrm{dom}(\varphi)\}.$$

Let $\tau : Bl_0\mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ be the regular map that is induced by the projection $\mathbb{A}^{n+1} \times \mathbb{P}^n \to \mathbb{A}^{n+1}$ to the first factor.
Compute all fibres of $\tau : Bl_0\mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ and determine their dimensions.
Remark: $Bl_0\mathbb{A}^{n+1}$ is called the blow-up of $\mathbb{A}^{n+1}$ in 0. The regular map $\tau : Bl_0\mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ is called the blow-down map.

Exercise 3. (4 points) *Degenerations of hypersurfaces*
Let $f, g \in k[x_0, x_1, \ldots, x_{n+1}]$ be homogeneous polynomials of degree d. Consider

$$X := V_{\mathbb{P}^{n+1}\times\mathbb{A}^1}(f + t(g - f)) \subset \mathbb{P}^{n+1} \times \mathbb{A}^1,$$

where t denotes the coordinate function on $\mathbb{A}^1$.

(a) Show that X is a quasi-projective variety if f and g are coprime.

(b) Consider the regular map $p : X \to \mathbb{A}^1$ that is induced by the projection $\mathbb{P}^{n+1} \times \mathbb{A}^1 \to \mathbb{A}^1$ to the second factor. Compute the fibres $X_0 := p^{-1}(0)$ and $X_1 := p^{-1}(1)$ of p above 0 and 1.

(c) Specialize to the case where $g = \ell_1 \ldots \ell_d$ is a product of d linear homogeneous polynomials ℓ_i that are pairwise coprime. Conclude that for any given natural number $m \geq 1$, there are regular maps between quasi-projective varieties such that one fibre is irreducible while another one has m many irreducible components.

(d) Specialize to the case where $g = \ell^d$ is a power of a linear homogeneous polynomial ℓ. Conclude that there are regular maps between quasi-projective varieties such that one fibre is isomorphic any given hypersurface $V(f) \subset \mathbb{P}^{n+1}$, while another fibre is isomorphic to $\mathbb{P}^n$.

Hand in: before noon on Monday, November 26th in the appropriate box on the 1st floor.