Wintersemester 2018/19

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Algebraic Geometry

Sheet 5

Unless specified otherwise, we will always work over an algebraically closed field k.

Exercise 1. (4 points) *The dimension of the affine cone*
Let $I \subset k[x_0, \ldots, x_n]$ be a homogeneous ideal. Let $X := V_{\mathbb{P}^n}(I) \subset \mathbb{P}^n$ be the corresponding projective variety and let $Y := V_{\mathbb{A}^{n+1}}(I) \subset \mathbb{A}^{n+1}$. Assume that X is non-empty. Show that

$$\dim X + 1 = \dim Y.$$

Exercise 2. (4 points) *Hypersurfaces*
Let $X \subset \mathbb{P}^n_k$ be a projective variety of dimension $n-1$. Show that X is a hypersurface, i.e. $X = V(F)$ for a non-constant irreducible homogeneous polynomial $F \in k[x_0, \ldots, x_n]$.

Exercise 3. (8 points) *Products*
Let n, m be natural numbers. Consider $\mathbb{P}^{(n+1)(m+1)-1}$ with homogeneous coordinates z_{ij} with $0 \leq i \leq n$ and $0 \leq j \leq m$. Consider the map of sets

$$\phi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}, \quad (x, y) \mapsto \phi((x, y)),$$

where the (i, j)-th coordinate of $\phi(x, y)$ is given by $z_{ij}(\phi(x, y)) = x_i y_j$. That is, ϕ is given by

$$\phi((x, y)) = [x_0y_0 : x_1y_0 : \cdots : x_ny_0 : x_0y_1 : x_1y_1 : \cdots : x_ny_1 : \cdots : x_0y_m : x_1y_m : \cdots : x_ny_m].$$

(a) Show that ϕ is injective.

(b) Show that the image of ϕ is a projective variety.

(c) Let $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ be projective varieties. Show that $\phi(X \times Y) \subset \mathbb{P}^{(n+1)(m+1)-1}$ is a projective variety.

Remark: From now on, we use ϕ to put the structure of an algebraic variety on the product $X \times Y$ of any two projective varieties X and Y. In particular, we obtain the structure of a projective variety on the product $\mathbb{P}^n \times \mathbb{P}^m$.

(d) Show that $\mathbb{P}^n \times \mathbb{P}^m$ can be covered by open subsets that are isomorphic to $\mathbb{A}^{n+m}$.

Hand in: before noon on Monday, November 19th in the appropriate box on the 1st floor.