Wintersemester 2018/19

Prof. Dr. Stefan Schreieder
Dr. Feng Hao

Algebraic Geometry

Sheet 4

Unless specified otherwise, we will always work over an algebraically closed field k.

Exercise 1. (4 points) *Quadrics are rational*
Let $n \geq 2$ and let $f \in k[x_1, \ldots, x_n]$ be an irreducible polynomial of degree two and consider its vanishing locus $X := V(f) \subset \mathbb{A}^n_k$. Show that there is a birational map $\phi : X \dashrightarrow \mathbb{A}^{n-1}_k$.
Hint: You have seen the case $n = 2$ in class.

Exercise 2. (4 points) *A cubic curve that is irrational*
Assume that k has characteristic different from 2 and 3. Consider the irreducibel polynomial

$$F := x_1^3 + x_2^3 - 1 \in k[x_1, x_2]$$

and let $X := V(F) \subset \mathbb{A}^2$. In this exercise you show that X is not rational, i.e. it is not birational to $\mathbb{A}^1$. The proof goes by contradiction, and so we assume for a contradiction that X is rational.

(a) Show that there are non-constant polynomials $f, g, h \in k[t]$ with

$$f^3 + g^3 = h^3.$$

(b) Show that in item (a) you may assume that f, g and h have pairwise no common factor, i.e. $\gcd(f, g) = 1$, $\gcd(f, h) = 1$ and $\gcd(g, h) = 1$. Show further that up to relabelling (and possibly multiplication by a third root of unity), you may assume that $\deg f \geq \deg g \geq \deg h$.

(c) Consider the formal partial derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial g}{\partial t}$ and $\frac{\partial h}{\partial t}$ and show that

$$f^2 h \frac{\partial f}{\partial t} + g^2 h \frac{\partial g}{\partial t} = (f^3 + g^3) \frac{\partial h}{\partial t}.$$

(d) Conclude that f^2 divides $g \frac{\partial h}{\partial t} - h \frac{\partial g}{\partial t}$ and derive a contradiction by degree considerations.

Remark: Note that the argument immediately generalizes to higher degree equations, to yield the following result: Let $d \geq 3$ be an integer and assume that the characteristic of k is either zero or larger than d. Then the affine variety

$$X := V(x_1^d + x_2^d - 1) \subset \mathbb{A}^2_k$$

is not rational.

Exercise 3. (4 points) *Basic projective geometry*
A projective variety $L \subset \mathbb{P}^2_k$ is called a line if it is cut out by a single homogeneous polynomial $\ell \in k[x_0, x_1, x_2]$ of degree one: $L = V(\ell) \subset \mathbb{P}^2_k$.

(a) Prove that any two lines in $\mathbb{P}^2_k$ have nontrivial intersection.

(b) Let $F \in k[x_0, x_1, x_2]$ be a non-constant homogeneous polynomial. Show that any line in $\mathbb{P}^2_k$ has nontrivial intersection with $X := V_{\mathbb{P}^2_k}(F) \subset \mathbb{P}^2_k$.

Exercise 4. (4 points) *Conics are isomorphic to* $\mathbb{P}^1$
Consider $X := V(t_0t_1 - t_2^2) \subset \mathbb{P}^2_k$. Show that

$$\phi : \mathbb{P}^1_k \to X, \quad [x_0 : x_1] \mapsto [x_0^2 : x_1^2 : x_0x_1]$$

is a regular map that admits a regular inverse.

Hand in: before noon on Monday, November 12th in the appropriate box on the 1st floor.