Wintersemester 2018/19

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Algebraic Geometry

Sheet 3

Unless specified otherwise, we will always work over an algebraically closed field k.

Exercise 1. (4 points) *Birational maps come from isomorphisms of function fields*
Let $\phi : X \dashrightarrow Y$ be a rational map between affine algebraic varieties. Show that ϕ is birational if and only if $\phi^* : k(Y) \to k(X)$ is an isomorphism of fields.

Exercise 2. (4 points) *Some basic facts about rational maps.*
Let $\varphi : X \dashrightarrow Y$ be a rational map between affine algebraic varieties.

(a) Let $U := \mathrm{dom}(\varphi)$ be the domain of φ. Show that $\varphi|_U : U \to Y$ is continuous.

(b) Let $\psi : X \dashrightarrow Y$ be another rational map. Suppose that there is a non-empty open subset $V \subset \mathrm{dom}(\varphi) \cap \mathrm{dom}(\psi)$ such that $\varphi|_V = \psi|_V$. Show that $\varphi = \psi$; that is, if $Y \subset \mathbb{A}^m_k$, $\varphi = (\varphi_1, \dots, \varphi_m)$ and $\psi = (\psi_1, \dots, \psi_m)$ for some rational functions $\varphi_i, \psi_i \in k(X)$, then $\varphi_i = \psi_i$ for all i.

Exercise 3. (4 points) *Standard open subsets.*
Let $X \subset \mathbb{A}^n_k$ be an affine algebraic set. Open subsets of X of the form $U_f := X \setminus V_X(f)$ for some $f \in k[X]$ are called standard open subsets of X.

(a) Let $f \in k[X]$ be non-zero. Show that $U_f \subset X$ is isomorphic to the affine algebraic set
$$Y \subset \mathbb{A}^n_k \times \mathbb{A}^1_k,$$
cut out by the ideal that is generated by $I(X)$ together with the function $t \cdot F - 1$, where $F \in k[x_1, \dots, x_n]$ is a polynomial with $F|_X = f$ and where t denotes a coordinate function on $\mathbb{A}^1_k$.

(b) In the notation of part (a), show that $k[U_f]$ is isomorphic to the localization $k[X]_f$ of $k[X]$ at the multiplicative set $\{1, f, f^2, f^3, \dots\}$.

Exercise 4. (4 points) *Regular functions on $\mathbb{A}^2_k \setminus \{0\}$.*
Consider the quasi-affine algebraic variety $X := \mathbb{A}^2_k \setminus \{0\}$ and let $i : X \to \mathbb{A}^2_k$ be the inclusion. Show that the pullback map
$$i^* : k[\mathbb{A}^2_k] \to k[X]$$
is an isomorphism.

Hand in: before noon on Monday, November 5th in the appropriate box on the 1st floor.