SHEET 5

Ex3 (c).

Denote the homogeneous coordinates of $\mathbb{P}^{(n+1)(m+1)-1}$ to be $[z_{0,0} : z_{1,0} : \ldots : z_{n,0} : z_{0,1} : \ldots : z_{n,1} : \ldots : z_{0,m} : \ldots : z_{n,m}]$.

Let $X = V_{\mathbb{P}^n}(I)$ and $Y = V_{\mathbb{P}^m}(J)$ for I, J being homogeneous ideals.

Claim: $\underline{\phi(X \times Y) = V_{\mathbb{P}^{(n+1)(m+1)-1}}(P)}$, where the homogeneous ideal P, is generated by the following homogeneous polynomials:

$$\{z_{i,j} \cdot z_{k,l} - z_{i,l} \cdot z_{k,j} \mid 0 \le i, k \le n; 0 \le j, l \le m\}$$

$$\{f(z_{0,j}, \ldots, z_{n,j}) \mid 0 \le j \le m, f \in I \text{ a homogeneous polynomial}\}$$

$$\{g(z_{i,0}, \ldots, z_{i,m}) \mid 0 \le i \le n, g \in J \text{ a homogeneous polynomial}\}.$$

Proof of the claim: "$\subseteq$" is clear, which is similar to the argument in (b).

"$\supseteq$". For any point $z = [\{z_{i,j}\}] \in V(P)$. As that in (b), we can assume that $z_{i_0,j_0} \neq 0$ for some i_0, j_0. Then there is a unique point

$$([\frac{z_{0,j_0}}{z_{i_0,j_0}} : \ldots : \frac{z_{n,j_0}}{z_{i_0,j_0}}], [\frac{z_{i_0,0}}{z_{i_0,j_0}} : \ldots : \frac{z_{i_0,m}}{z_{i_0,j_0}}])$$

in $\mathbb{P}^n \times \mathbb{P}^m$ that maps to z through ϕ by (a). Also, note that

$$[\frac{z_{0,j_0}}{z_{i_0,j_0}} : \ldots : \frac{z_{n,j_0}}{z_{i_0,j_0}}] \in X, \ [\frac{z_{i_0,0}}{z_{i_0,j_0}} : \ldots : \frac{z_{i_0,m}}{z_{i_0,j_0}}] \in Y,$$

since $f(z_{0,j_0}, \ldots, z_{n,j_0}) = 0$ for any homogeneous polynomial $f \in I$, and $g(z_{i_0,0}, \ldots, z_{i_0,m}) = 0$ for any homogeneous polynomial $g \in J$.

Now it remains to show that $\phi(X \times Y)$ is irreducible. Thus it suffices to show that $X \times Y$ is irreducible.

Assume BY CONTRADICTION that $X \times Y = Z \cup Z'$ with Z and Z' two closed proper subset such that $Z \not\subseteq Z'$ and $Z' \not\subseteq Z$. Consider the following two open subsets

$$U = \{y \in Y \mid X \times \{y\} \subseteq Z\}; \ U' = \{y \in Y \mid X \times \{y\} \subseteq Z'\}.$$

Note that $Y = U \cup U'$. Since Y is irreducible so WLOG we can assume U is dense in Y. Claim that the Zariski closure $\overline{X \times U}$ is actually $X \times Y$. In fact, note that the closed subset of $X \times Y$ is given by the pullback of the closed subset of $\phi(X \times Y) \subset \mathbb{P}^{(n+1)(m+1)-1}$. Any closed subset $\phi(X \times Y)$ is defined by homogeneous equations $h(z_{0,0} : \ldots : z_{n,m}) = 0$. Plugging $z_{i,j} = x_i y_j$ into h, one get a bihomogeneous polynomial $H([x_0 : \ldots : x_n], [y_0 : \ldots : y_m])$ in $\{x_i\}$ and $\{y_i\}$. Thus any closed subset of $X \times Y$ is the vanishing locus of some bihomogeneous polynomials H. Now consider any bihomogeneous polynomial H. If it vanishes on $X \times U$, then it vanishes over the whole $X \times Y$ by continuity (Reason: Take any point $(a, b) \in X \times Y$. Consider any bihomogeneous equation H. Then $H(a, y)$ vanishes on the dense open subset $\{a\} \times U \subset \{a\} \times Y$. Hence $H(a, b) = 0$.). Hence $\overline{X \times U} = X \times Y$. Thus $X \times Y \subseteq Z$. Then we get a contradiction, which implies $X \times Y$ is irreducible with the topology induced from ϕ.

(d) As described in the remark, we transport the topology from the image $\phi(\mathbb{P}^n \times \mathbb{P}^m)$ (which is inherited from the topology of the ambient space $\mathbb{P}^{(n+1)(m+1)}$) to the product $\mathbb{P}^n \times \mathbb{P}^m$. Thus it suffices to show that the image $\phi(\mathbb{P}^n \times \mathbb{P}^m)$ can be covered by open subsets that are isomorphic to $\mathbb{A}^{n+m}$.

Thus it suffices to show that for $U_i := \{x_i \neq 0\} \subset \mathbb{P}^n$, $V_j := \{y_j \neq 0\} \subset \mathbb{P}^m$, $\phi(U_i \times V_j) \cong \mathbb{A}^{n+m}$. WLOG, we consider U_0, V_0, and the following maps:

$$\Psi : \phi(U_0 \times V_0) \to \mathbb{A}^{n+m};\ \phi([1 : x_1 : \ldots : x_n], [1 : y_1 : \ldots : y_m]) \mapsto (x_1, \ldots, x_n, y_1 \ldots y_m)$$

and

$$\Phi : \mathbb{A}^{n+m} \to \phi(U_0 \times V_0);\ (x_1, \ldots, x_n, y_1 \ldots y_m) \mapsto [1 : x_1 : \ldots : x_n : y_1 : x_1y_1 : \ldots : x_ny_1 : \ldots : x_ny_m].$$

Note that Ψ and Φ are regular maps, and

$$\phi([1 : x_1 : \ldots : x_n], [1 : y_1 : \ldots : y_m]) = [1 : x_1 : \ldots : x_n : y_1 : x_1y_1 : \ldots : x_ny_1 : \ldots : x_ny_m]$$

Thus Ψ and Φ are inverse to each other. Hence we have the isomorphism $\phi(U_i \times V_j) \cong \mathbb{A}^{n+m}$.