Ex 1. (a) Let $F = x_1^6 + x_2^6 - x_0^4 x_1 x_2$
(1) If char$k \neq 2, 3$, we have the partial differentials of F

$$\frac{\partial F}{\partial x_0} = -4x_0^3 x_1 x_2; \ \frac{\partial F}{\partial x_1} = 6x_1^5 - x_0^4 x_2; \ \frac{\partial F}{\partial x_2} = 6x_2^5 - x_0^4 x_1.$$

Let $F = \frac{\partial F}{\partial x_i} = 0$. Note first $x_0 \neq 0$, otherwise, $x_1 = x_2 = 0$. Hence we get $x_0 \neq 0$ and $x_1 = x_2 = 0$. Thus in this case the singular point is $[1 : 0 : 0]$.
(2) If char$k = 2$, we have the partial differentials of F

$$\frac{\partial F}{\partial x_0} = 0; \ \frac{\partial F}{\partial x_1} = -x_0^4 x_2; \ \frac{\partial F}{\partial x_2} = -x_0^4 x_1.$$

Let $F = \frac{\partial F}{\partial x_i} = 0$. Hence we get $x_1^6 + x_2^6 = 0$, i.e., $x_1 = \xi x_2$ with $\xi^6 = 1$. If $x_0 \neq 0$, we have $x_1 = x_2 = 0$. If $x_0 = 0$, then we have $x_1 \neq 0$ and $x_1 = \xi x_2$ with $\xi^6 = 1$. Hence the singular points are $[1 : 0 : 0]$ and $\{[0 : \xi : 1] \mid \xi^6 = 1\}$.
(3) If char$k = 3$, we have the partial differentials of F

$$\frac{\partial F}{\partial x_0} = 2x_0^3 x_1 x_2; \ \frac{\partial F}{\partial x_1} = -x_0^4 x_2; \ \frac{\partial F}{\partial x_2} = -x_0^4 x_1.$$

Let $F = \frac{\partial F}{\partial x_i} = 0$. Hence we get $x_1^6 + x_2^6 = 0$, i.e., $x_1 = \xi x_2$ with $\xi^6 = 1$. If $x_0 \neq 0$, we have $x_1 = x_2 = 0$. If $x_0 = 0$, then we have $x_1 \neq 0$ and $x_1 = \xi x_2$ with $\xi^6 = 1$. Hence the singular points are $[1 : 0 : 0]$ and $\{[0 : \xi : 1] \mid \xi^6 = 1\}$.

(b) Let $F = x_1^4 + x_2^4 - x_1^3 x_0$
(1) If char$k \neq 2, 3$, we have the partial differentials of F

$$\frac{\partial F}{\partial x_0} = -x_1^3; \ \frac{\partial F}{\partial x_1} = 4x_1^3 - 3x_1^2 x_0; \ \frac{\partial F}{\partial x_2} = 4x_2^3.$$

Let $F = \frac{\partial F}{\partial x_i} = 0$. Note first $x_0 \neq 0$, otherwise, $x_1 = x_2 = 0$. Hence we get $x_0 \neq 0$ and $x_1 = x_2 = 0$. Thus in this case the singular point is $[1 : 0 : 0]$.
(2) If char$k = 2$, we have the partial differentials of F

$$\frac{\partial F}{\partial x_0} = -x_1^3; \ \frac{\partial F}{\partial x_1} = -3x_1^2 x_0; \ \frac{\partial F}{\partial x_2} = 0.$$

Let $F = \frac{\partial F}{\partial x_i} = 0$. Hence we get $x_1 = x_2 = 0$. Hence the singular point is just $[1 : 0 : 0]$.
(3) If char$k = 3$, we have the partial differentials of F

$$\frac{\partial F}{\partial x_0} = -x_1^3; \ \frac{\partial F}{\partial x_1} = 4x_1^3; \ \frac{\partial F}{\partial x_2} = 4x_2^3.$$

Let $F = \frac{\partial F}{\partial x_i} = 0$. Hence we get $x_1 = x_2 = 0$. Hence the singular points is just $[1 : 0 : 0]$.

(c) Let $F = x_0^3 + x_3^3 + x_0 x_2 x_3$
(1) If char$k \neq 3$, we have the partial differentials of F

$$\frac{\partial F}{\partial x_0} = 3x_0^2 + x_2 x_3; \ \frac{\partial F}{\partial x_1} = 0; \ \frac{\partial F}{\partial x_2} = x_0 x_3; \ \frac{\partial F}{\partial x_3} = 3x_3^2 + x_0 x_2.$$

Let $F = \frac{\partial F}{\partial x_i} = 0$. Note that x_0 must be zero. Then $x_3 = 0$ and x_1, x_2 cannot be both 0. If $x_0 \neq 0$. Hence the singular locus are $[0 : x_1 : x_2 : 0]$.
(2) If char$k = 3$, we have the partial differentials of F

$$\frac{\partial F}{\partial x_0} = x_2 x_3; \ \frac{\partial F}{\partial x_1} = 0; \ \frac{\partial F}{\partial x_2} = x_0 x_3; \ \frac{\partial F}{\partial x_3} = x_0 x_2.$$

Let $F = \frac{\partial F}{\partial x_i} = 0$. Note that x_0 must be zero. Then $x_3 = 0$. Hence the singular locus are $[0 : x_1 : x_2 : 0]$.

Ex 4. (a) This is a local problem (the Zariski tangent space can be defined intrinsically), we can assume $X \subset \mathbb{A}^n$ is an affine variety with coordinate functions $\{x_i\}$, $Y \subset \mathbb{A}^m$ is an affine variety with coordinate functions $\{y_j\}$, and $f : X \to Y$ can be written into $f(x) = (f_1(x), \dots, f_m(x))$. Now take any element $G \in I(Y)$, we have $G(f_1(x), \dots, f_m(x)) = 0$ for any $x \in X$. Hence the composition function $G(f_1, \dots, f_m) \in I(X)$. Now we fix any closed point $a \in X$. WLOG we can assume that $a = (0, \dots, 0) \in \mathbb{A}^n$ and $f(a) = (0, \dots, 0) \in \mathbb{A}^m$. Thus for any tangent vector $v = (v_1, \dots, v_n) \in T_{X,a}$, we have

$$\sum_{k=1}^{n} \frac{\partial G(f)}{\partial x_k}(a) v_k = 0.$$

Hence we get

$$\sum_{j=1}^{m} \frac{\partial G}{\partial y_j}(f(a)) \cdot (\sum_{k=1}^{n} \frac{\partial f_j}{\partial x_k}(a) v_k) = 0,$$

i.e., $d_{f(a)}G(d_a f_1(v), \dots, d_a f_m(v)) = 0$ for any tangent vector $v \in T_{X,a}$. Then we get $(d_a f_1(v), \dots, d_a f_m(v)) \in T_{Y,f(a)}$ for any $v \in T_{X,a}$. Thus f induces a linear map $d_a f : T_{X,a} \to T_{Y,f(a)}$.

(b) Consider the regular map $f : \mathbb{A}^1 \to Y := V_{\mathbb{A}^2}(x_2^2 - x_1^3)$, given by $t \mapsto (t^2, t^3)$, where t is the coordinate function of $\mathbb{A}^1$.
(1) If $a = 0 \in \mathbb{A}^1$, then $d_a f = [2t, 3t^2]|_{t=0} = [0, 0]$. Hence $d_a f$ is of rank 0.
(2) If $a \neq 0$ in $\mathbb{A}^1$, then $d_a f = [2t, 3t^2]|_{t=a} = [2a, 3a^2]$. Hence $d_a f$ is of rank 1.