Exercise 2: For the second part of the this exercise, we want to construct the fibre product $\mathbb{P}^1 \times \mathbb{P}^1$. Cover left hand side $\mathbb{P}^1$ by two affine lines $\operatorname{Spec} k[u]$ and $\operatorname{Spec} k[v]$ and right hand side $\mathbb{P}^1$ by two affine lines $\operatorname{Spec} k[s]$ and $\operatorname{Spec} k[t]$. As what we did in the first part, we take two steps. Firstly, we have two copies of affine planes $U_1 = \operatorname{Spec} k[u] \times \operatorname{Spec} k[t] \cong \mathbb{A}^2$, and $U_2 = \operatorname{Spec} k[u] \times \operatorname{Spec} k[s] \cong \mathbb{A}^2$, with two open subsets $\operatorname{Spec} k[u] \times \operatorname{Spec} k[t, t^{-1}]$, and $\operatorname{Spec} k[u] \times \operatorname{Spec} k[s, s^{-1}]$, respectively, with two inclusions induced by natural morphisms of commutative rings $k[u, t] \to k[u, t, t^{-1}]$ and $k[u, s] \to k[u, s, s^{-1}]$. Then we glue U_1 and U_2 through the isomorphism $\phi_1 : k[u, t, t^{-1}] \to k[u, s, s^{-1}]; t \mapsto s^{-1}$. Then we get a scheme $V_1 = \operatorname{Spec} k[u] \times \mathbb{P}^1$. Similarly we get $V_2 = \operatorname{Spec} k[v] \times \mathbb{P}^1$. As the second step, we consider open subsets $\operatorname{Spec} k[u, u^{-1}] \times \mathbb{P}^1 \subset \operatorname{Spec} k[u] \times \mathbb{P}^1$ and $\operatorname{Spec} k[v, v^{-1}] \times \mathbb{P}^1 \subset \operatorname{Spec} k[v] \times \mathbb{P}^1$ with two inclusions induced by universal property of fibre product and natural morphisms of commutative rings $k[u] \to k[u, u^{-1}]$ and $k[v] \to k[v, v^{-1}]$. Then glue V_1 and V_2 through the isomorphism $\phi_2 : \operatorname{Spec} k[u, u^{-1}] \times \mathbb{P}^1 \to \operatorname{Spec} k[v, v^{-1}] \times \mathbb{P}^1; u \mapsto v^{-1}$. At the end, we get the fibre product $\mathbb{P}^1 \times \mathbb{P}^1$. It is reduced and irreducible, since we can cover it by $\mathbb{A}^2$, which is dense, reduced and irreducible.

Exercise 4: (c) By Ex1 of Sheet 4, we know that the k-rational point of $\operatorname{Spec} k[x]$ are of type $(x - a)$ for $a \in k$. The scheme theoretic fiber of the morphism

$$f : \operatorname{Spec} k[x, y]/(xy) \to \operatorname{Spec} k[x]$$

over the point $p = (x - a)$ is $f^{-1}(p) = \frac{k[x]}{(x-a)} \otimes \frac{k[x,y]}{(xy)}$, where we note that $\frac{k[x]}{(x-a)} = k(p)$ the residue field of p. Hence $f^{-1}(p) = \frac{k[x,y]}{(xy,x-a)} = \frac{k[y]}{(ay)}$.

1) If $a \neq 0$,i.e., a is a unit, then $\frac{k[y]}{(ay)} \cong k$. Hence $f^{-1}(p)$ is $\operatorname{Spec} k$, which is irreducible, reduced, and connected.

2) If $a = 0$, then $\frac{k[y]}{(ay)} \cong k[y]$. Hence $f^{-1}(p)$ is $\operatorname{Spec} k[y]$, which is irreducible, reduced, and connected.