Exercise 4: (a) Closed immersion $f : X \to Y$ is a morphism of finite type.
Proof: Since f is a closed immersion, any point $y \in Y$, and an affine open subset $V = SpecB \subset Y$, the morphism

$$f : f^{-1}(V) \to V$$

is also a closed immersion. By HW sheet 6 EX2, we have the above morphism is induced by $B \to B/I$. Since B/I is finitely generated B-algebra, so we are done.

(b) An open immersion which is quasi-compact is of finite type.
Proof: For any point $y \in Y$, we can take an open NBHD $V = \operatorname{Spec} B$ of y in Y. Then $f^{-1}(V) = X \cap V$ is open in $\operatorname{Spec} B$. Then we can cover $X \cap V$ by open subset $\{\operatorname{Spec} B_f |\ f \in B\}$. Since $f : X \to Y$ is quasi-compact, $X \cap V$ can be covered by finitely many $\operatorname{Spec} B_{f_i}$, for $i = 1, \dots m$. Also, note that B_{f_i} are finitely generated B-algebra. Then we are done.

(c) Let $f : X \to Y$, $g : Y \to Z$ are of finite type. Then for any $z \in Z$, there is an open NBHD of $W = \operatorname{Spec} C \subset Z$ of z such that $g^{-1}(W)$ can be covered by open affine subsets $g^{-1}(W) = \bigcup_{i=1}^{m} V_i$ with $V_i = \operatorname{Spec} B_i$ and B_i are f.g. C-algebras. By Ex1, each $f^{-1}(V_i)$ can be covered by open affine subsets $f^{-1}(V_i) = \bigcup_j \operatorname{Spec} A_{ij}$ with A_{ij} are finitely generated B_i-algebra. Also, note that since f is of finite type, f is quasi-compact. Thus we can assume $f^{-1}(V_i) = \bigcup_{j=1}^{n} \operatorname{Spec} A_{ij}$. Then $(g \circ f)^{-1}(W) = \bigcup_{ij} \operatorname{Spec} A_{ij}$ and A_{ij} are f.g. C-algebra.

(d) First we claim that for any $f : X \to Y$ is of finite type, any open affine subset $V = \operatorname{Spec} B \subset Y$ and any open affine subset $U = \operatorname{Spec} A \subset f^{-1}(V)$, then A is f.g. B-algebra.
Proof of the claim: By Ex1, $f^{-1}(V) = \bigcup_i U_i$ with $U_i = \operatorname{Spec} A_i$ and A_i being f.g. B-algebra. Note that each open set $U_i \cap V$ can be covered by affine open subsets $\operatorname{Spec} A_{f_k} \cong \operatorname{Spec} A_{i g_\alpha}$. Since A_i are f.g. B-algebra, $A_{i g_\alpha}$ hence A_{f_k} are f.g. B-algebra. Also, since U is quasi-compact, there exist $f_1, \dots, f_s$ s.t. $U = \bigcup_{i=1}^{s} A_{f_i}$. Hence A is a finitely generated B-algebra.
Now we go back to the proof of (d). Consider $f : X \to Y$ is quasi-compact and $g : Y \to Z$ is any morphism, such that $g \circ f$ is of finite type. Now we want to show that f is of fintie type. For any point $y \in Y$ and $z := g(y) \in Z$, we take an open NBHD of z say $W = \operatorname{Spec} C \subset Z$. Take $V := \operatorname{Spec} B \subset g^{-1}(W)$ with $y \in V$ open NBHD and a cover $f^{-1}(V) = \bigcup_i \operatorname{Spec} A_i$. Then by previous discussion, A_i are finitely generated C-algebras through the morphism $C \to B \to A_i$. Hence A_i are fintiely generated B-algberas. Then f is locally of finitely type, hence of finitely type since f is quasi-compact.