Exercise 3: b)
For the scheme $(X, \mathcal{O}_X)$, take an open affine cover $\{X_i = \operatorname{Spec} A_i\}_{i\in I}$ of X. Then since X_i are subschemes of X, we have natural gluing data:
• $X_{ij} := X_i \cap X_j$, with structure sheaf $\mathcal{O}_X|_{X_{ij}} = \mathcal{O}_{\operatorname{Spec} A_i}|_{X_{ij}}$.
• Isomorpism between the open subschemes $(\varphi_{ij}, \varphi_{ij}^\sharp) : (X_{ji}, \mathcal{O}_X|_{X_{ji}}) \to (X_{ij}, \mathcal{O}_X|_{X_{ij}})$. Here on the topological spaces we have $\varphi_{ij} = id$.
• Those data satisfies: $(\varphi_{ii}, \varphi_{ii}^\sharp) = id_{X_i}$; $(\varphi_{ij}, \varphi_{ij}^\sharp) = (\varphi_{ji}, \varphi_{ji}^\sharp)^{-1}$; and $(\varphi_{ki}, \varphi_{ki}^\sharp) \circ (\varphi_{ij}, \varphi_{ij}^\sharp) = (\varphi_{kj}, \varphi_{kj}^\sharp)$.
In the previous tutorial class, we have already seen that the projection morhpism

$$p_i : A_i \to A_i^{\text{red}} := A_i / \operatorname{nil} A_i$$

induces an homeomorphism

$$(f_i, f_i^\sharp) : \operatorname{Spec} A_i^{\text{red}} \cong \operatorname{Spec} A_i = X_i$$

as topologcial spaces.
Now we consider the following data:
• Define the affine scheme $X_i^{\text{red}} := \operatorname{Spec} A_i^{\text{red}}$.
• Now we define the scheme X_{ij}^{red}. The underlying topological space is defined to be the open subset $f_i^{-1}(X_{ij})$ and the structure sheaf is defined to be $\mathcal{O}_{\operatorname{Spec} A_i^{\text{red}}}|_{f_i^{-1}(X_{ij})}$.
• We construct the gluing isomorphism

$$(\varphi_{ij}^{\text{red}}, \varphi_{ij}^{\text{red}\,\sharp}) : (X_{ji}^{\text{red}}, \mathcal{O}_{X_{ji}^{\text{red}}}) \to (X_{ij}^{\text{red}}, \mathcal{O}_{X_{ij}^{\text{red}}})$$

where $\varphi_{ij}^{\text{red}}$ is defined to be the continuous map making the following diagram commutative

$$\begin{tikzcd} X_{ji} \arrow[r, "\varphi_{ij}=Id"] & X_{ij} \\ X_{ji}^{\text{red}} \arrow[u, "f_j", "\cong"'] \arrow[r, "\varphi_{ij}^{\text{red}}"] & X_{ij}^{\text{red}} \arrow[u, "\cong", "f_i"'] \end{tikzcd} ,$$

i.e., $\varphi_{ij}^{\text{red}} = f_i^{-1} \circ f_j$. Also, the sheaf morphism $\varphi_{ij}^{\text{red}\,\sharp}$ is defined as follows: for any open set $U \subset X_{ij}^{\text{red}}$, and any section $s \in \mathcal{O}_{X_{ij}^{\text{red}}}(U) = \mathcal{O}_{\operatorname{Spec} A_i^{\text{red}}}(U)$, we can take a cover $U = \bigcup_\alpha W_\alpha$ of U, such that $s = \{[s_\alpha, W_\alpha]\}$, and s_α can be lift to elements $\tilde{s}_\alpha$ in $\mathcal{O}_{\operatorname{Spec} A_i}(f_i^{-1}(W_\alpha))$. Then $\varphi_{ij}^{\text{red}\,\sharp}$ is defined to be

$$\varphi_{ij}^{\text{red}\,\sharp}(s) := \{[f_j^\sharp \circ \varphi_{ij}^\sharp(\tilde{s}_\alpha), \varphi_{ji}^{\text{red}}(W_\alpha)]\}.$$

This map is a well defined map, since two different lifting of s_α are upto a nilpotent element, and $f_j^\sharp$ kills the nilpotent element. And for the same reason, $\{[f_j^\sharp \circ \varphi_{ij}^\sharp(\tilde{s}_\alpha), \varphi_{ji}^{\text{red}}(W_\alpha)]\}$ is glued to a section in $\mathcal{O}_{\operatorname{Spec} A_j}(\varphi_{ji}^{\text{red}}(U))$. Similarly we can construct the morphism $\varphi_{ji}^\sharp$. Localize everything at $x \in X_{ij}^{\text{red}}$ (which corresponds to a prime ideal $P \subset A_i$ and $P' \subset A_j$ via homeomorphisms f_i and $f_j \circ \varphi_{ji}^{\text{red}}$, we also denote their image in the reduced ring A_i^{red} and A_j^{red} to be $\bar{P}$, $\bar{P}'$, resp.) we get the following diagram

$$\begin{tikzcd} \ker \arrow[r, "\cong"] \arrow[d] & \ker \arrow[d] \\ A_{i,P} \arrow[r, "\cong"] \arrow[d] & A_{j,P'} \arrow[d] \\ A_{i,\bar{P}}^{\text{red}} \arrow[r] & A_{j,\bar{P}'}^{\text{red}} \end{tikzcd}$$

which implies $(\varphi_{ij}^{\text{red}}, \varphi_{ij}^{\text{red}\,\sharp})$ are isomorphisms. It is clear that we can check the three cocycle condtions $(\varphi_{ii}^{\text{red}}, \varphi_{ii}^{\text{red}\,\sharp}) = id_{X_i}$; $(\varphi_{ij}^{\text{red}}, \varphi_{ij}^{\text{red}\,\sharp}) = (\varphi_{ji}^{\text{red}}, \varphi_{ji}^{\text{red}\,\sharp})^{-1}$; and $(\varphi_{ki}^{\text{red}}, \varphi_{ki}^{\text{red}\,\sharp}) \circ (\varphi_{ij}^{\text{red}}, \varphi_{ij}^{\text{red}\,\sharp}) = (\varphi_{kj}^{\text{red}}, \varphi_{kj}^{\text{red}\,\sharp})$ by reducing them to the cocycle conditions for $\{(\varphi_{ij}, \varphi_{ij}^\sharp)\}$, since the ambiguities are nilpotents. With the new constructed gluing data, we get a scheme $(X^{\mathrm{red}}, \mathcal{O}_{X^{\mathrm{red}}})$ and morphism $(j, j^\sharp) : (X^{\mathrm{red}}, \mathcal{O}_{X^{\mathrm{red}}}) \to (X, \mathcal{O}_X)$, which is induced by $(f_i, f_i^\sharp)$, since $(f_i, f_i^\sharp)$ is compatible with the gluing data.

c) When Y is affine, say $Y = \operatorname{Spec} A$, let $p : A \to A^{\mathrm{red}} := A/\operatorname{nil} A$ be the projection (corresponds to $\tau : \operatorname{Spec} A^{\mathrm{red}} \to \operatorname{Spec} A$). We have a homomorphism $\varphi : A \to \mathcal{O}_X(X)$ induced by f. Since $\ker p = \operatorname{nil} A \subset \ker \varphi$, we have a unique homorphism $\phi : A^{\mathrm{red}} \to \mathcal{O}_X(X)$. Hence we get a morphism $(g, g^\sharp) : (X, \mathcal{O}_X) \to (Y^{\mathrm{red}}, \mathcal{O}_{Y^{\mathrm{red}}})$, where $g := \tau^{-1} \circ f$ on the topological space, and $g^\sharp : \mathcal{O}_{Y^{\mathrm{red}}} \to g_* \mathcal{O}_X$ is induced by ϕ. The uniqueness follows from the uniqueness of the morphism ϕ.
In general, we consider the morphsim $f : X \to Y$ between two schemes with X reduced.Then on the underlying topological spaces we have the morphism $g := j^{-1} \circ f$. Now we need to define $g^\sharp : \mathcal{O}_{Y^{\mathrm{red}}} \to g_* \mathcal{O}_X$. Just as in the proof of b), we take affine open cover $\{Y_i := \operatorname{Spec} A_i\}_{i \in I}$ of Y, then we get a covering $X_i := f^{-1}(Y_i)$ of X. By the previous discussion, we have morphisms

$$(g_i, g_i^\sharp) : (X_i, \mathcal{O}_{X_i}) \to (\operatorname{Spec} A_i^{\mathrm{red}}, \mathcal{O}_{\operatorname{Spec} A_i^{\mathrm{red}}})$$

Note that $\{(g_i, g_i^\sharp)\}$ commute with gluing morphisms, since $\tau_i : \operatorname{Spec} A_i^{\mathrm{red}} \to \operatorname{Spec} A_i$ are compatible with the gluing data for Y^{red} as described in b). Thus $(g_i, g_i^\sharp)$ are glued to a morphism $(g, g^\sharp)$, which is unique by the uniqueness of $(g_i, g_i^\sharp)$.

Exercise 4: 1) Since φ preserves the grading, the ideal $< \varphi(S_+) >$ generated by $\varphi(S_+)$ is a homogeneous ideal. Thus $U := \{P \in \operatorname{Proj} T |\ P \not\supset \varphi(S_+)\} = \operatorname{Proj} T - V_{\operatorname{Proj} T}(< \varphi(S_+) >)$, which is an open set.

2) For any point $P \in U$, there exists a homogeneous element $f \in S_+$ s.t., $P \notin V(\varphi(f))$. As usual, we denote $\operatorname{Proj} T - V(\varphi(f))$ by $U^+_{\varphi(f)}$. Then $P \in U^+_{\varphi(f)} \subset U$. Then U is covered by standard open sets. Note that $\varphi : S \to T$ induces

$$\varphi_{(f)} : S_{(f)} \to T_{\varphi(f)}; \frac{s}{f^m} \mapsto \frac{\varphi(s)}{\varphi(f)^m}.$$

This impies the morphism of schemes $\psi_f : U^+_{\varphi(f)} \to U^+_f \subset \operatorname{Proj} S$. For the overlap

$$U^+_{\varphi(f)} \cap U^+_{\varphi(g)} = \operatorname{Proj} T - V(\varphi(fg)) = U^+_{\varphi(fg)}$$

we have the following two morphisms

$$S_{(f)} \to (S_{(f)})_{\frac{g^{\deg f}}{f^{\deg g}}} \cong S_{(fg)} \to T_{\varphi(fg)}$$

and

$$S_{(g)} \to (S_{(g)})_{\frac{f^{\deg g}}{g^{\deg f}}} \cong S_{(gf)} \to T_{\varphi(gf)}$$

which implies two morphisms of schemes

$$\psi_f|_{U^+_{\varphi(fg)}} : U^+_{\varphi(fg)} \to \operatorname{Spec} S_{(f)} \subset \operatorname{Proj} S$$

and

$$\psi_g|_{U^+_{\varphi(gf)}} : U^+_{\varphi(gf)} \to \operatorname{Spec} S_{(g)} \subset \operatorname{Proj} S.$$

The above two morphisms are the coincide with each other since the above two algebra maps $S_{(f)} \to S_{(fg)}$ and $S_{(g)} \to S_{(gf)}$ induce the incusions and the $S_{(fg)} \to T_{\varphi(fg)}$ and $S_{(gf)} \to T_{\varphi(gf)}$ are the same map induced by φ. Thus we can glue the scheme morphisms ψ_f to get a natural morphism $\psi : U \to \operatorname{Proj} S$.