## EX SHEET 12

Ex4. (b) We want to show that the sheaf $\mathcal{E}$ of regular sections of vector bundle $\pi : E \to X$ is an $\mathcal{O}_X$-module. To fixed the notation, let $(U_i, \varphi_i)$ be the local trivialization in the definite of $\pi$. For any open set, $s \in \mathcal{E}(U)$ and $f \in \mathcal{O}_X(U)$, we first define action $f \cdot s$. By sheaf condition we just need to define $f|_{U\cap U_i} \cdot s|_{U\cap U_i}$ over $U \cap U_i$, and check that $(f|_{U\cap U_i} \cdot s|_{U\cap U_i})|_{U_i\cap U_j\cap U} = (f|_{U\cap U_j} \cdot s|_{U\cap U_j})|_{U_i\cap U_j\cap U}$.

By the commutative diagram

$$\begin{tikzcd}
\pi^{-1}(U\cap U_i) \arrow[r, shift left, "\varphi_i"] & (U\cap U_i)\times k^r \arrow[l, shift left, "\varphi_i^{-1}"] \\
U\cap U_i \arrow[u, "s"] \arrow[ur, bend right] &
\end{tikzcd}$$

we define

$$f|_{U\cap U_i} \cdot s|_{U\cap U_i} := \varphi_i^{-1} \circ (f \cdot \varphi_i \circ s|_{U\cap U_i}).$$

Notice that.

$$\begin{aligned}
(f|_{U\cap U_i} \cdot s|_{U\cap U_i})|_{U_i\cap U_j\cap U} =& \varphi_i^{-1} \circ (f \cdot \varphi_i \circ s|_{U\cap U_i\cap U_j}) \\
=& \varphi_j^{-1} \circ \varphi_j \circ \varphi_i^{-1}(f \cdot \varphi_i \circ \varphi_j^{-1} \circ \varphi_j \circ s|_{U\cap U_i\cap U_j}) \\
=& \varphi_j^{-1} \circ (f \cdot \varphi_j \circ s|_{U\cap U_i\cap U_j}) \\
=& (f|_{U\cap U_j} \cdot s|_{U\cap U_j})|_{U_i\cap U_j\cap U}.
\end{aligned}$$

Thus by the sheaf condition for $\mathcal{E}$, we have $f \cdot s$ is a well defined section in $\mathcal{E}(U)$. It is trivial that with this action, $\mathcal{E}(U)$ is a $\mathcal{O}_X(U)$-module.

(c) To show $\mathcal{E}$ is locally free, we just need to show that over each $U_i$, we have the isomorphism of $\mathcal{O}_{U_i}$-module isomorphism

$$\mathcal{E}|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus r}$$

In fact, over each $V \subset U_i$ we define the morphism between $\mathcal{O}_{U_i}$-modules

$$\begin{aligned}
\Phi_i(V) : \mathcal{E}|_{U_i}(V) &\longrightarrow \mathcal{O}_{U_i}(V)^{\oplus r} \\
s &\longmapsto \varphi_i \circ s
\end{aligned}$$

Note that since

$$\Phi_i(V)(f \cdot s) = \varphi_i \circ f \cdot s = f(\varphi_i \circ s) = f\Phi_i(V)(s),$$

*Date*: January 23, 2019.

$\varphi_i$ is an $\mathcal{O}_{U_i}(V)$-module morphism. Note also since $\varphi_i$ is an ismorphism, $\Phi_i(V)$ is an isomorphism between two $\mathcal{O}_{U_i}(V)$-modules for any open set $V \subset U_i$. Thus

$$\mathcal{E}|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus r}.$$

(d)(Remark) Given any lcally free sheaf $\mathcal{E}$ of rank $r$, we want to construct the corresponding algebraic vector bundle $E$. Cover $X = \bigcup U_i$ such that there is an isomorphism of $\mathcal{O}_{U_i}$-modules $\phi_i : \mathcal{E}|_{U_i} \to \mathcal{O}_{U_i}^{\oplus r}$. Note that since $\phi_i$ are $\mathcal{O}_{U_i}$-module isomorphism, we have over $U_i \cap U_j$, $\phi_j \circ \phi_i^{-1} : \mathcal{O}_{U_i \cap U_j}^{\oplus r} \to \mathcal{O}_{U_i \cap U_j}^{\oplus r}$ is an isomorphism of free $\mathcal{O}_{U_i \cap U_j}$-modules, i.e., $\phi_j \circ \phi_i^{-1}$ is an invertible matrix of regular function over $U_i \cap U_j$. Also, with these gluing data, we get an quasi-projective variety $E$ out of open covers $U_i \times \mathbb{A}_k^r$.