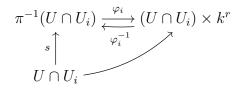
## EX SHEET 12

Ex4. (b) We want to show that the sheaf  $\mathcal{E}$  of regular sections of vector bundle  $\pi: E \to X$  is an  $\mathcal{O}_X$ -module. To fixed the notation, let  $(U_i, \varphi_i)$  be the local trivialization in the definite of  $\pi$ . For any open set,  $s \in \mathcal{E}(U)$  and  $f \in \mathcal{O}_X(U)$ , we first define action  $f \cdot s$ . By sheaf condition we just need to define  $f|_{U \cap U_i} \cdot s|_{U \cap U_i}$  over  $U \cap U_i$ , and check that  $(f|_{U \cap U_i} \cdot s|_{U \cap U_i})|_{U_i \cap U_j \cap U} = (f|_{U \cap U_j} \cdot s|_{U \cap U_j})|_{U_i \cap U_j \cap U}$ .

By the commutative diagram



we define

$$f|_{U\cap U_i} \cdot s|_{U\cap U_i} := \varphi_i^{-1} \circ (f \cdot \varphi_i \circ s|_{U\cap U_i}).$$

Notice that.

$$(f|_{U\cap U_i} \cdot s|_{U\cap U_i})|_{U_i \cap U_j \cap U} = \varphi_i^{-1} \circ (f \cdot \varphi_i \circ s|_{U\cap U_i \cap U_j})$$
$$= \varphi_j^{-1} \circ \varphi_j \circ \varphi_i^{-1} (f \cdot \varphi_i \circ \varphi_j^{-1} \circ \varphi_j \circ s|_{U\cap U_i \cap U_j})$$
$$= \varphi_j^{-1} \circ (f \cdot \varphi_j \circ s|_{U\cap U_i \cap U_j})$$
$$= (f|_{U\cap U_i} \cdot s|_{U\cap U_i})|_{U_i \cap U_i \cap U_i}.$$

Thus by the sheaf condition for  $\mathcal{E}$ , we have  $f \cdot s$  is a well defined section in  $\mathcal{E}(U)$ . It is trivial that with this action,  $\mathcal{E}(U)$  is a  $\mathcal{O}_X(U)$ -module.

(c) To show  $\mathcal{E}$  is locally free, we just need to show that over each  $U_i$ , we have the isomorphism of  $\mathcal{O}_{U_i}$ -module isomorphism

$$\mathcal{E}|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus r}$$

In fact, over each  $V \subset U_i$  we define the morphism between  $\mathcal{O}_{U_i}$ -modules

$$\Phi_i(V) : \mathcal{E}|_{U_i}(V) \longrightarrow \mathcal{O}_{U_i}(V)^{\oplus i}$$
$$s \longmapsto \varphi_i \circ s$$

Note that since

$$\Phi_i(V)(f \cdot s) = \varphi_i \circ f \cdot s = f(\varphi_i \circ s) = f\Phi_i(V)(s),$$

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## EX SHEET 12

 $\varphi_i$  is an  $\mathcal{O}_{U_i}(V)$ -module morphism. Note also since  $\varphi_i$  is an isomorphism,  $\Phi_i(V)$  is an isomorphism between two  $\mathcal{O}_{U_i}(V)$ -modules for any open set  $V \subset U_i$ . Thus

$$\mathcal{E}|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus r}$$

(d)(Remark) Given any lcally free sheaf  $\mathcal{E}$  of rank r, we want to construct the corresponding algebraic vector bundle E. Cover  $X = \bigcup U_i$  such that there is an isomorphism of  $\mathcal{O}_{U_i}$ -modules  $\phi_i : \mathcal{E}|_{U_i} \to \mathcal{O}_{U_i}^{\oplus r}$ . Note that since  $\phi_i$  are  $\mathcal{O}_{U_i}$ -module isomorphism, we have over  $U_i \cap U_j$ ,  $\phi_j \circ \phi_i^{-1} : \mathcal{O}_{U_i \cap U_j}^{\oplus r} \to \mathcal{O}_{U_i \cap U_j}^{\oplus r}$  is an isomorphism of free  $\mathcal{O}_{U_i \cap U_j}$ -modules, i.e.,  $\phi_j \circ \phi_i^{-1}$  is an invertible matrix of regular function over  $U_i \cap U_j$ . Also, with these gluing data, we get an quasi-projective variety E out of open covers  $U_i \times \mathbb{A}_k^r$ .