EX SHEET 11

Proof of Ex4. d) Assume that there is an isomorphism

$$\psi : \mathcal{E} :\to \mathcal{C}^0_{S^1,\mathbb{R}^1}.$$

It suffices to construct an homeomorphism ϕ between vector bundles E and $S^1 \times \mathbb{R}^1$ from ψ, and ϕ maps zero sections s_0 of E to zero section $X \times \{0\}$ of $X \times \mathbb{R}^1$, since we get the contradiction by connectedness of $E - s_0$ and $X \times (\mathbb{R}^1 - \{0\})$.

First we fix the notations. As is introduced in the exercise, the local trivialization (homeomorphism)

$$\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^1,$$

where $\pi : E \to X$ is the nature projection. For each $x \in U_i$, φ_i defines real vector space structure on $\pi^{-1}(x)$ through the bijection $\pi^{-1}(x) \to \{x\} \times \mathbb{R}^1$. Now we can define the zero section s_0 of E. Now we recall the $\mathcal{C}^0_{S^1,\mathbb{R}^1}$-module structure of $\mathcal{E}$. For open set $U \subset X, f \in \mathcal{C}^0_{S^1,\mathbb{R}^1}(U), s \in \mathcal{E}(U)$, $f \cdot s$ is defined by the local sections $f \cdot s|_{U_i \cap U} := \varphi_i^{-1} \circ (f\varphi_i) \circ s|_{U_i \cap U}$, and it is easy to check the those sections can be glued by checking

$$\varphi_j^{-1} \circ (f\varphi_j \circ s|_{U_j}) = \varphi_i^{-1} \circ \varphi_i \circ \varphi_j^{-1} \circ (f\varphi_j \circ \varphi_i^{-1} \circ \varphi_i \circ s|_{U_j}) = \varphi_i^{-1} \circ (f\varphi_i \circ s|_{U_i})$$

over $U_i \cap U_j$.

Define

$$\phi : E \to X \times \mathbb{R}^1$$
$$p \mapsto (x, \psi_U(s)(x)),$$

where $p \in E$, with $\pi(p) = x$. $s : U \to \pi^{-1}(U)$ is a local section such that $s(x) = p$. Now claim that ϕ is well defined. In fact, for $p \notin s_0$, if there are two local sections s_1 and s_2 on U such that $s_1(p) = s_2(p)$. Then we can assume there is $f \in \mathcal{C}^0_{S^1,\mathbb{R}^1}(U)$ such that $s_2 = fs_1$ by shrinking U if necessary and $f(p) = 1$. Then

$$\psi_U(s_2)(x) = \psi_U(f \cdot s_1)(x) = f(x)\psi_U(s_1)(x) = \psi_U(s_1)(x).$$

If $p \in s_0$, $\psi_U(s)(p) = 0$ for any s, since ψ is assumed to be $\mathcal{C}^0_{S^1,\mathbb{R}^1}$-module isomorphism.

Firstly, since ψ is $\mathcal{C}^0_{S^1,\mathbb{R}^1}$-module isomorphism, ϕ_x is an linear isomorphism. In particular, ϕ is a bijection. Secondly, claim that ϕ is a homeomorphism. In fact, Using local trivialization φ_i, we get the morphism

$$\phi|_{U_i} \circ \varphi_i^{-1} : U_i \times \mathbb{R}^1 \to U_i \times \mathbb{R}^1$$
$$(x, v) \mapsto (x, \psi_{U_i}(\varphi_i^{-1} \circ v \cdot s)(x)),$$

where $s : U_i \to U_i \times \mathbb{R}^1$ is any section such that $s(x) = 1$. Since $\varphi_i^{-1} \circ v \cdot s)$ is a continuous section of E over U_i, $\psi_{U_i}(\varphi_i^{-1} \circ v \cdot s)$ is a continuous section of $U_i \times \mathbb{R}^1$. Hence $\phi|_{U_i} \circ \varphi_i^{-1}$ is continuous. Similarly, we can show that $\varphi_i^{-1} \circ \phi|_{U_i}$ is continuous. Note also by construction ϕ maps s_0 to $X \times \{0\}$.