SHEET 8

Ex3. When char k does not divide d, we take the example $F = x_0^d + \ldots + x_{n+1}^d$, where $n \geq 1$.

First we calculate the partial derivatives:

$$\frac{\partial F}{\partial x_i} = d \cdot x_i^{d-1}.$$

Then for any point $[a_0 : \ldots : a_{n+1}] \in X := V_{\mathbb{P}^{n+1}}(F)$, there exist an $0 \leq i \leq n+1$, such that $\frac{\partial F}{\partial x_i}(a_0, \ldots, a_{n+1}) \neq 0$. Hence X is smooth.

Then we need to show that F is irreducible. Use Eisenstein's Criterion for polynomial $F = x_0^d + \ldots + x_{n+1}^d \in k[x_1, \ldots, x_{n+1}][x_0]$. It suffices to show that $F_0 = x_1^d + \ldots + x_{n+1}^d \in k[x_1, \ldots, x_{n+1}]$ is irreducible. Thus by induction, we need to show that $x^d + y^d + z^d \in k[x, y, z]$ is irreducible. Note that since $(y+z) | y^d + z^d$, to show $x^d + y^d + z^d$ irreducible, again by Eisenstein Criterion, we just need to show that $(y+z)^2 \nmid (y^d + z^d)$.

Assume that

$$y^d + z^d = (y^2 + 2yz + z^2)(a_{d-2}y^{d-2} + a_{d-3}y^{d-3}z + \ldots + a_1 yz^{d-3} + a_0 z^{d-2}).$$

Expanding it and comparing the coefficients of both side, we get $a_0 = 1, a_1 = -2$, and

$$a_{d-i} + 2a_{d-i-1} + a_{d-i-2} = 0.$$

Hence the general term formula is

$$a_n = (-1)^n(n+1).$$

however, we also get $a_{d-2} = 1$ by comparing the coefficients, which is a contradiction. Hence F is irreducible.

1) When char $k = 2$, n is odd and $d = 2$, we can just take

$$F = \sum_{i_1 \neq i_2} x_{i_1} x_{i_2}.$$

2) When char $k = 2$, $n = 2m$ is even and $d = 2$, we can take

$$F = \sum_{i=0}^{m} x_i x_{m+1+i}.$$

3) In general, when $d > 2$ and char k divides d, we can take the so called Gabber's hypersurface

$$F = x_0^d + \sum_{i=0}^{n} x_i x_{i+1}^{d-1}.$$