Exercise 3:
For the scheme $(Y, \mathcal{O}_Y)$ and the closed subscheme Z, take an open affine cover $\{Y_i = \operatorname{Spec} A_i\}_{i \in I}$ of X. Let $Z_i := Z \cap Y_i$ be the closed subset of Y_i. By Ex2, we can assume that Z_i are underlying topological spaces of the scheme $(Z_i, \mathcal{O}_{Z_i} := \mathcal{O}_{SpecA_i/I_i})$, where $I_i \subset A_i$ is an ideal with $I_i = \sqrt{I_i}$. Especially, Z_i are reduced. We denote the closed imbedding

$$\phi_i : Z_i \hookrightarrow Y_i.$$

The we have the following data:
- $Y_{ij} := Y_i \cap Y_j$, with structure sheaf $(\Psi_i, \Psi_i^\sharp) = \mathrm{Id} : Y_{ij} \to Y_{ji}$, with trivial gluing data for Y.
- $Z_{ij} := Z_i \cap Y_{ij}$ as topological space and $\mathcal{O}_{Z_{ij}} := \mathcal{O}_{Z_i|_{Z_{ij}}}$.
- Now we construct the gluing isomorphism

$$(\varphi_{ij}, \varphi_{ij}^\sharp) : (Z_{ji}, \mathcal{O}_{Z_{ji}}) \to (Z_{ij}, \mathcal{O}_{Z_{ij}})$$

where $\varphi_{ij} = \mathrm{Id}$ and we have the following commutative diagram

$$\begin{array}{ccc} X_{ji} & \xrightarrow{\Psi_{ij}=\mathrm{Id}} & X_{ij} \\ \phi_j \uparrow & & \uparrow \phi_i \\ Z_{ji} & \xrightarrow{\varphi_{ij}=\mathrm{Id}} & Z_{ij} \end{array} .$$

The sheaf morphism $\varphi_{ij}^\sharp : \mathcal{O}_{Z_{ij}} \to \varphi_{ij*}\mathcal{O}_{Z_{ji}}$ is defined as follows: for any open set $U \subset Z_{ij}$, and any section $s \in \mathcal{O}_{Z_{ij}}(U)$, since Z_i are closed subscheme of X_i, we can take a covering $U = \bigcup_\alpha W_\alpha$ of U, such that $s = \{[s_\alpha, W_\alpha]\}$, and s_α can be lift to elements $\tilde{s}_\alpha$ in $\mathcal{O}_{X_i}(V_\alpha)$, where V_α is open in X_i and $V_\alpha \cap Z = W_\alpha$. Then $\varphi_{ij}^\sharp$ is defined to be

$$\varphi_{ij}^\sharp(s) := \{[\Psi_{ij}^\sharp(\tilde{s}_\alpha), W_\alpha)]\},$$

where the equivalent class is get from the surjective map $\phi_j^\sharp : \mathcal{O}_{X_j} \to \phi_{j*}\mathcal{O}_{Z_j}$. This map is a well defined map, since two different lifting of s_α are upto a element evaluating zero at Z, and it will be killed after taking direct limit to get a section on $W_\alpha \subset Z_j$. And for the same reason, $\{[\Psi_{ij}^\sharp(\tilde{s}_\alpha), W_\alpha)]\}$ is glued to a section in $\mathcal{O}_{Z_{ji}}(U)$. Then we get the morphism $\varphi_{ij}^\sharp$, Similarly we can construct the morphism $\varphi_{ji}^\sharp$.Easy to check that $(\varphi_{ij}, \varphi_{ij}^\sharp)$ and $(\varphi_{ji}, \varphi_{ji}^\sharp)$ are the inverse to each other. It is clear that we can check the three cocycle condtions $(\varphi_{ii}, \varphi_{ii}^\sharp) = id_{X_i}$; $(\varphi_{ij}, \varphi_{ij}^\sharp) = (\varphi_{ji}, \varphi_{ji}^\sharp)^{-1}$; and $(\varphi_{ki}, \varphi_{ki}^\sharp) \circ (\varphi_{ij}, \varphi_{ij}^\sharp) = (\varphi_{kj}, \varphi_{kj}^\sharp)$ by reducing them to the cocycle conditions for $\{(\Psi_{ij}, \Psi_{ij}^\sharp)\}$, since the ambiguities envaluating zero along Z. With the new constructed gluing data, we get a scheme $(Z, \mathcal{O}_Z)$ and morphism $(j, j^\sharp) : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$, which is induced by $(\phi_i, \phi_i^\sharp)$, since $(\phi_i, \phi_i^\sharp)$ is compatible with the gluing data, i.e., $\phi_i|_{Z_{ij}} = \phi_j|_{Z_{ji}}$. $(Z, \mathcal{O}_Z)$ is reduced since we can reduce it to each affine piece $(Z_i, \mathcal{O}_{Z_i})$, which is reduced by construction.

Ex 4: "$\subseteq$" For all $a \in \mathfrak{a} \subset A$, then $\frac{a}{1} \in \mathfrak{a}A_{f_i}$ and $\varphi_i^{-1}(\frac{a}{1}) = a$.
"$\supseteq$" For any $a \in \bigcap_{i=1}^n \varphi_i^{-1}(\mathfrak{a}A_{f_i})$, $\varphi_i(a) = \frac{a}{1} \in \mathfrak{a}A_{f_i}$ for all i. Thus there exist $b_i \in \mathfrak{a}$ and $m_i\mathbb{Z}^{\geq 0}$ such that $\frac{a}{1} = \frac{b_i}{f_i^{m_i}}$ in A_{f_i}. Thus there exists n_i s.t. $f_i^{n_i}(af_i^{m_i} - b_i) = 0$ in A for all i. We can choose m and n large enough such that $f_i^n(af_i^m - b) = 0$ in A for all i. Especially, $f_i^{m+n}a \in \mathfrak{a}$ for all i.
Since $\bigcup_{i=1}^n U_{f_i} = \bigcup_{i=1}^n (X - V(f_i)) = X$, we have $\bigcap_{i=1}^n V(f_i) = \emptyset = V(\overset{n}{\underset{i=1}{}} < f_i >)$. This implies there exist $t_i \in A$ s.t. $1 = \sum_{i=1}^n t_i f_i$. Now we take large enough integer N such that $1 = (\sum_{i=1}^n t_i f_i)^N = \sum_k F_k(f_1, \ldots, f_n)$, with each monomial containing at least one f_i of power great than $n+m$. Thus we have $a = a \cdot \sum_k F_k(f_1, \ldots, f_n) \in \mathfrak{a}$.