LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

Sommersemester 2019

Prof. Dr. Stefan Schreieder
Dr. Feng Hao

Algebraic Geometry 2

Sheet 8

Exercise 1. (4 points) *Comparison of fibre products: schemes versus topology*
Let $f : X \to S$ and $g : Y \to S$ be morphisms of affine schemes $X = \operatorname{Spec} B_1$, $Y = \operatorname{Spec} B_2$ and $S = \operatorname{Spec} A$. There is a natural continuous map

$$\Phi : |X \times_S X| \to |X| \times_{|S|} |Y|$$

from the topological space which underlies the fibre product $X \times_S Y$ in the category of schemes, to the fibre product $|X| \times_{|S|} |Y|$ of the underlying toppological spaces $|X|$, $|Y|$ and $|S|$ in the category of topological spaces. Show that the fibre of Φ above $(x, y) \in |X| \times_{|S|} |Y|$ is given by the topological space

$$|\operatorname{Spec} \kappa(x) \times_{\operatorname{Spec} \kappa(s)} \operatorname{Spec} \kappa(y)|,$$

where $s = f(x) = f(y) \in S$.

Exercise 2. (4 points) *Two self-products*
Let k be a field. Consider the schemes X_1 and X_2 over k from Exercise 4 on sheet 4, obtained by gluing two copies of $\mathbb{A}^1_k$ along $\mathbb{A}^1_k \setminus \{0\}$ with respect to the ring isomorphism $k[t, t^{-1}] \to k[s, s^{-1}]$, given by $t \mapsto s$, respectively $t \mapsto s^{-1}$. Describe in each case the fibre product $X_i \times_{\operatorname{Spec} k} X_i$ via gluing appropriate affine schemes and decide whether it is reduced and/or irreducible. Try also to describe the underlying topological space.

Exercise 3. (4 points) *Fibre products of morphisms of finite type*
Let $f : X \to Y$ be a morphism of finite type. Show that for any morphism of schemes $S \to Y$, the natural morphism $X \times_Y S \to S$ is of finite type. Conclude in particular that the fibre X_y of f above a point $y \in Y$ is a scheme of finite type over the field $\kappa(y)$.

Exercise 4. (4 points) *Fibres of morphisms*
Let k be a field. Compute the fibres of the following morphisms above all k-rational points and decide in each case which fibres are irreducible, reduced, respectively connected.

(a) $f : \operatorname{Spec} k[x] \to \operatorname{Spec} k[x]$, induced by $k[x] \to k[x]$, $x \mapsto x^2$;

(b) $f : \operatorname{Spec} k[x, y, t]/(xy - t) \to \operatorname{Spec} k[t]$, induced by the natural map $k[t] \to k[x, y, t]/(xy - t)$;

(c) $f : \operatorname{Spec} k[x, y]/(xy) \to \operatorname{Spec} k[x]$, induced by the natural map $k[x] \to k[x, y]/(xy)$.

Hand in: before noon on Monday, June 21th in the appropriate box on the 1st floor.