Sommersemester 2019

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Algebraic Geometry 2

Sheet 2

Exercise 1. (4 points) *Structure sheaf*
Let A be a ring and let $\mathcal{O}$ be the structure sheaf on $\operatorname{Spec} A$.

(a) Show that the stalk of $\mathcal{O}$ at $\mathfrak{p}$ is isomorphic to $A_{\mathfrak{p}}$;

Hint: The argument has been sketched in the lecture; you are supposed to give the details here.

(b) Show that for any $f \in A$,

$$\mathcal{O}(U_f) = A_f,$$

where $U_f := \operatorname{Spec} A \backslash V(f)$ and A_f is the localization of A at the multiplicative set $\{1, f, f^2, f^3, \dots\}$.

Hint: Generalize the proof of the corresponding statement for the sheaf of regular functions of a variety that you have seen in class last term. Note that in that proof, we used that our rings were automatically Noetherian last term. You are not allowed to assume this here. Instead you should find a small argument which shows that you can make the required finiteness assumption also in the case where A might not be Noetherian.

Exercise 2. (4 points) *Local rings and local homomorphisms*
Recall that a ring A is a local ring if it has a unique maximal ideal $\mathfrak{m}_A$. A homomorphism between local rings $\varphi : A \to B$ is called a local homomorphism if $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B$.

(a) Show that a ring A is a local ring if and only if $\operatorname{Spec} A$ has a unique closed point.

(b) Show that a homomorphism $\varphi : A \to B$ between local rings is local if and only if the induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ maps the closed point of $\operatorname{Spec} B$ to the closed point of $\operatorname{Spec} A$.

(c) Give an example of a homomorphism $\varphi : A \to B$ between local rings which is not local. Describe the induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ in your example.

(d) Give examples of local rings A, such that $\operatorname{Spec} A$ has exactly

  (i) one point;

  (ii) two points;

  (iii) infinitely many points.

Exercise 3. (4 points) *Spectrum of a discrete valuation ring*
Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$, residue field $k = R/\mathfrak{m}$ and fraction field $K = \operatorname{Frac} R$.

(a) Describe the topological space $\operatorname{Spec} R$.

(b) Compute the structure sheaf $\mathcal{O}$ on any open subset of $\operatorname{Spec} R$.

(c) Compute all stalks of the structure sheaf $\mathcal{O}$ on $\operatorname{Spec} R$.

Remark: Important examples of discrete valuation rings are given by localizations $A_\mathfrak{p}$ of normal rings A at hight one prime ideals $\mathfrak{p}$; you have seen this last term where $A = k[X]$ for a normal affine curve A and $\mathfrak{p}$ is the ideal of a point $x \in X$. If $X = \mathbb{A}^1$ and $x = 0 \in \mathbb{A}^1$, then the corresponding local ring is given by the localization $k[t]_{(t)}$. Other important examples of discrete valuation rings are given by the power series ring $k[[t]]$ over any field k, or the p-adic integers $\mathbb{Z}_p$ for some prime p.

Exercise 4. (4 points) *Functions on* $\operatorname{Spec} A$.

(a) Give an example of a ring A and a non-zero element $f \in \Gamma(\operatorname{Spec} A, \mathcal{O})$, such that for all $\mathfrak{p} \in \operatorname{Spec} A$, the value of f in the residue field $\kappa(\mathfrak{p})$ vanishes.

(b) Can you give an example as above in the case where $A = k[X]$ for an affine variety X over an algebraically closed field k? (If not, try to pin down the ring theoretic property that $k[X]$ has which excludes a phenomenon as in (a) above.)

Hand in: before noon on Monday, May 6th in the appropriate box on the 1st floor.