LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATICHES INSTITUT

Sommersemester 2019 Prof. Dr. Stefan Schreieder
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Algebraic Geometry 2

Sheet 12

Exercise 1. (4 points) *Injective objects in the category of abelian groups.*
Show that an abelian group G is an injective object in the category of abelian groups if and only if it is divisible, which means that for any positive integer n, multiplication with n yields a surjection $[n] : G \to G$.

Exercise 2. (4 points) *Ext groups*
Let $\mathcal{A}b$ be the category of abelian groups. Recall that for any abelian group G, the functor $\operatorname{Hom}(G, -) : \mathcal{A}b \to \mathcal{A}b$ is a covariant left exact functor. By Exercise 2, applied to $R = \mathbb{Z}$, the category of abelian groups has enough injective objects. We may therefore define the right derived functors of $\operatorname{Hom}(G, -)$; we denote them by

$$\operatorname{Ext}^i(G, -) := R^i \operatorname{Hom}(G, -).$$

(a) Show that $\operatorname{Ext}^i(G, \mathbb{Z}) = 0$ and $\operatorname{Ext}^i(G, \mathbb{Z}/2) = 0$ for all $i \geq 2$ and all abelian groups G.

(**Hint:** Find an injective resolutions of $\mathbb{Z}$ and $\mathbb{Z}/2$ of suitable length.)

(b) Compute $\operatorname{Ext}^i(\mathbb{Z}, \mathbb{Z}/2)$ for $i = 0, 1$.

(c) Compute $\operatorname{Ext}^i(\mathbb{Z}/2, \mathbb{Z})$ for $i = 0, 1$.

Exercise 3. (4 points) *Extensions of abelian groups*
For any abelian groups (or R-modules) A and B, we consider the set of extensions of A by B, i.e. short exact sequences $0 \to B \to E \to A \to 0$, where E is some abelian group. We denote by $\operatorname{Ext}(A, B)$ the set of such extensions modulo the equivalence relation, where we say that two extensions $0 \to B \to E \to A \to 0$ and $0 \to B \to E' \to A \to 0$ are equivalent if there is an isomorphism $E \cong E'$ of groups which is compatible with the identity on the subgroup B and the quotient group A, respectively.

(a) Show that any extension $0 \to B \to E \to A \to 0$ gives rise to an element $\varphi_E \in \operatorname{Ext}^1(A, B)$.

(**Hint:** Pick an injective resolution $B \to J^*$ of B and use it to compute $\operatorname{Ext}^1(A, B)$ explicitly.)

(b) Show that $\varphi_E = 0 \in \operatorname{Ext}^1(A, B)$ if and only if $E \cong A \oplus B$ is isomorphic to the trivial extension.

(c) One can show that the map

$$\operatorname{Ext}(A, B) \to \operatorname{Ext}^1(A, B), \quad E \mapsto \varphi_E$$

that you defined above is in fact a bijection. Use this fact (without proof) to deduce from Exercise 2 a complete description of all extensions (up to isomorphisms) of $\mathbb{Z}$ by $\mathbb{Z}/2$, and of $\mathbb{Z}/2$ by $\mathbb{Z}$.

Exercise 4. (6 points) *The category of R-modules has enough injectives*
Let R be a ring. The purpose of this exercise is to show that the category of R-modules has enough injective objects. We will break up the prove in several small steps.
For any R-module M, we define $M^\vee := \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ as the group of all group homomorphisms $M \to \mathbb{Q}/\mathbb{Z}$. Clearly, the R-module structure on M yields a canonical R-module structure on $M^\vee$.

(a) Show that for any R-module M, there is a natural isomorphism of R-modules

$$\mathrm{Hom}_R(M, R^\vee) \cong M^\vee.$$

(b) Show that $R^\vee$ is an injective object in the category of R-modules.

(**Hint:** for any inclusion of R-modules $M \subset N$, you need to show that the induced map $\mathrm{Hom}_R(N, R^\vee) \to \mathrm{Hom}_R(M, R^\vee)$ is surjective. To this end, you may use part (a) and Exercise 1.)

(c) Conclude from parts (a) and (b) that $M \mapsto M^\vee$ yields an exact contravariant functor from the category of R-modules to itself.

(**Hint:** You may use without proof that for any R-module N, $\mathrm{Hom}_R(-, N)$ is a contravariant left exact functor.)

(d) For any R-module M, define the free R-module

$$F(M) := \bigoplus_{m \in M} R \cdot [m]$$

with basis indexed by the elements of M. Consider the natural map $F(M) \twoheadrightarrow M$ and convince yourself that it is surjective. Conclude from (c) that the induced map $M^\vee \to F(M)^\vee$ is injective.

(e) Show that the evaluation map $M \to (M^\vee)^\vee$ is an injective morphism of R-modules. Conclude from part (d) that the composition $M \to (M^\vee)^\vee \to F(M^\vee)^\vee$ is an injective morphism of R-modules.

(f) Show that $F(M^\vee)^\vee \cong \prod_{m \in M^\vee} R^\vee$ is a direct product of copies of $R^\vee$ that are indexed by the elements of $M^\vee$. Conclude from part (b) that $F(M^\vee)^\vee$ is an injective object in the category of R-module. Conclude finally from part (e) that the category of R-modules has enough injectives.

Hand in: before noon on Monday, July 22nd in the appropriate box on the 1st floor.