LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

Sommersemester 2019

Prof. Dr. Stefan Schreieder
Dr. Feng Hao

# Algebraic Geometry 2

Sheet 11

**Exercise 1.** (4 points) *Weighted projective space.*
Let $k$ be a field and let $n$ and $d$ be a positive integers. Consider the graded ring $S := k[y, x_0, x_1, \ldots, x_n]$, where $y$ has weight $d$ and each $x_i$ has weight 1. Let $X = \operatorname{Proj} S$ and consider the sheaf

$$\mathcal{O}_X(m) := \widetilde{S(m)}.$$

(a) Show that $\mathcal{O}_X(m)$ is locally free if $d \mid m$.

(b) Show that $\mathcal{O}_X(1)$ is not locally free if $d > 1$.

(**Hint:** Show that $\mathcal{O}_X(1)$ is not free locally around $\mathfrak{p} := (x_0, x_1, \ldots, x_n) \in \operatorname{Proj} S$ by considering the stalk $\mathcal{O}_X(1)_{\mathfrak{p}}$.)

**Exercise 2.** (4 points) *Closed subscheme of weighted projective space*
Consider the graded ring $S := k[y, x_0, x_1, \ldots, x_n]$ from Exercise 1. Let $e$ be a positive integer and let $f \in k[x_0, x_1, \ldots, x_n]$ be homogeneous of degree $ed$. Consider the closed subscheme

$$i : Y := \operatorname{Proj}(S/(y^e - f)) \hookrightarrow X.$$

(a) Show that the inclusion $k[x_0, \ldots, x_n] \subset S$ induces a finite morphism

$$p : Y \longrightarrow \mathbb{P}^n_k = \operatorname{Proj} k[x_0, \ldots, x_n].$$

(b) Assume that $Y$ is integral. Show that $\deg(p) = e$, where $\deg(p)$ is defined as the degree of the field extension $\kappa(\eta_{\mathbb{P}^n_k}) \subset \kappa(\eta_Y)$ given by the residue fields of the generic points of $\mathbb{P}^n_k$ and $Y$.

(c) Show that $p^*\mathcal{O}_{\mathbb{P}^n_k}(1) \cong i^*\mathcal{O}_X(1)$. In particular, $i^*\mathcal{O}_X(1)$ is locally free of rank one.

(**Remark:** This does not contradict Exercise 1(b) above, as one can show that in fact $\mathcal{O}_X(1)$ is locally free away from the point $\mathfrak{p} := (x_0, x_1, \ldots, x_n) \in \operatorname{Proj} S$ and this point does not lie on $Y$.)

**Exercise 3.** (4 points) *Global sections of $\mathcal{O}(1)$ on weighted projective space*
Let $k$ be a field. Let $d_0 \leq d_1 \leq \cdots \leq d_n$ be a sequence of positive integers and let $S := k[z_0, \ldots z_n]$ be the graded $k$-algebra with $|z_i| = d_i$. Let $X := \operatorname{Proj} S$ and consider $\mathcal{O}_X(j) := \widetilde{S(j)}$ for some $j \in \mathbb{Z}$. Show that the natural map

$$S_j \longrightarrow \Gamma(X, \mathcal{O}_X(j)), \quad f \mapsto \left(\mathfrak{p} \mapsto \frac{f}{1} \in S(j)_{(\mathfrak{p})}\right)$$

is an isomorphism. Use this to compute $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(j))$ for all $j$, where $k$ denotes an arbitrary field.

**Exercise 4.** (4 points) $M \mapsto \widetilde{M}$ *might not be an embedding in the graded case.*
Give an example of a graded ring $S$ with $X = \operatorname{Proj} S \neq \emptyset$ and a graded $S$-module $M$ with $M_0 \neq 0$, such that $\widetilde{M} = 0$. Conclude that in general the natural map

$$M_0 \longrightarrow \Gamma(X, \widetilde{M})$$

does not need to be injective.

**Hand in:** before noon on Monday, July 15th in the appropriate box on the 1st floor.