Sommersemester 2019

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Algebraic Geometry 2

Sheet 6

Exercise 1. (4 points) *An interesting example of a non-noetherian scheme*
Let k be a field. Consider the ring

$$A := \frac{k[x][y_\lambda \mid \lambda \in k]}{\langle (x-\lambda)y_\lambda, y_\lambda^2 \mid \lambda \in k\rangle},$$

with associated affine scheme $X := \operatorname{Spec} A$.

(a) Show that the underlying topological space of X is noetherian.

(b) Show that all local rings of X are noetherian.

(c) Show that X is not a noetherian scheme if k is infinite.

Exercise 2. (4 points) *Closed subschemes*

(a) Let A be a ring. Show that there is a bijection between the ideals of A and the closed subschemes of $X = \operatorname{Spec} A$;

(**Hint:** A crucial step here is to prove first that any closed subscheme of an affine scheme is affine.)

(b) Find all closed subschemes of $X = \operatorname{Spec} A$, where $A = k[x, y]/(x^2, y^2)$ for a field k, and indicate which of these closed subschemes are contained in each other.

Exercise 3. (4 points) *Closed subsets have a unique reduced subscheme structure*
Let $(Y, \mathcal{O}_Y)$ be a scheme, and let $Z \subset Y$ be a closed subset. Show that there is a unique closed immersion

$$(f, f^\sharp) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y),$$

such that $f(X) = Z$ and $(X, \mathcal{O}_X)$ is reduced.

Exercise 4. (4 points) *A step in the proof of Proposition 7.13*
Let A be a ring, $f_1, \dots, f_n \in A$ such that $X = \operatorname{Spec} A = \bigcup_{i=1}^n U_{f_i}$ is covered by the standard open subsets $U_{f_i} = \operatorname{Spec} A_{f_i}$. Let $\varphi_i : A \to A_{f_i}$ be the canonical homomorphism.
Show that for any ideal $\mathfrak{a} \subset A$, we have

$$\mathfrak{a} = \bigcap_{i=1}^n \varphi_i^{-1}(\mathfrak{a} \cdot A_{f_i}).$$

Hand in: before noon on Monday, June 3rd in the appropriate box on the 1st floor.