Sommersemester 2019

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Algebraic Geometry 2

Sheet 5

Exercise 1. (4 points) *Projective space*
Let R be a ring and consider the graded ring $S := R[x_0, x_1, \ldots, x_n]$, where $\deg x_i = 1$ for all i. Show that $\mathbb{P}^n_R$ is naturally isomorphic to $\operatorname{Proj} S$.

Exercise 2. (4 points) *Proj*(S)
Let k be a field and consider the graded ring $R := k[x_0, x_1, \ldots, x_n]$, where $\deg x_i = 1$ for all i. Let $I \subset R$ be a homogeneous ideal which does not contain R_+. Denote the quotient ring R/I by S.

(a) Show that $\operatorname{Proj} S$ can be covered by $n+1$ affine schemes.

(b) Assume that $I = (f)$ is generated by a single homogeneous polynomial f. Describe the affine schemes from item (a) explicitly in this case.

Exercise 3. (4 points) *Reduced schemes*
A scheme $(X, \mathcal{O}_X)$ is reduced if for every open subset $U \subset X$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.

(a) Show that $(X, \mathcal{O}_X)$ is reduced if and only if for every $p \in X$, the local ring $\mathcal{O}_{X,p}$ has no nilpotent elements.

(b) Let $(X, \mathcal{O}_X)$ be a scheme. Show that there is a canonical reduced scheme $(X^{\mathrm{red}}, \mathcal{O}_{X^{\mathrm{red}}})$ with a morphism
$$(j, j^\sharp) : (X^{\mathrm{red}}, \mathcal{O}_{X^{\mathrm{red}}}) \to (X, \mathcal{O}_X)$$
such that the underlying topological spaces of X^{red} and X coincide and $j : X^{\mathrm{red}} \to X$ is the identity.

Hint: Realize X via a gluing data of affine schemes $\operatorname{Spec} A_i$ and show that this gluing data gives a caoncical gluing data for the affine schemes $\operatorname{Spec} A_i^{\mathrm{red}}$, where A_i^{red} denotes the quotient of A_i by its nilradical (i.e. the ideal of all nilpotent elements of A_i).

(c) Let $f : X \to Y$ be a morphism of schemes with X reduced. Show that f factors through a unique morphism $g : X \to Y^{\mathrm{red}}$.

Exercise 4. (4 points) *Morphisms induced by homomorphisms of graded rings*
Let $\varphi : S \to T$ be a homomorphism of graded rings which preserves the gradings. Show that

$$U := \{\mathfrak{p} \in \operatorname{Proj} T \mid \mathfrak{p} \not\supseteq \varphi(S_+)\}$$

is an open subset of $\operatorname{Proj} T$, and that φ induces a natural morphism of schemes $f : U \to \operatorname{Proj} S$.

Hand in: before noon on Monday, May 27th in the appropriate box on the 1st floor.