Sommersemester 2019

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Algebraic Geometry 2

Sheet 4

Exercise 1. (4 points) *k-rational points of affine space*
Let k be a field and let $x \in \mathbb{A}_k^n$ be a k-rational point. Show that x corresponds to the maximal ideal $(t_1 - x_1, \dots, t_n - x_n)$ for some $x_i \in k$. Conclude that there is a bijection between the k-rational points of $\mathbb{A}_k^n$ and the elements of k^n.
Hint: Prove the case $n = 1$ first. For the general case, you might want to construct and use a morphism of affine schemes $\pi_i : \mathbb{A}_k^n \to \mathbb{A}_k^1$, which on k-rational points corresponds to projection to the i-th coordinate.

Exercise 2. (4 points) *Products of rings*
Let A be a ring. Assume that $e_1, \dots, e_n \in A$ is a complete set of idempotents of A, meaning that $e_i^2 = e_i$ for all i, $e_i e_j = 0$ for all $i \neq j$ and $1 = e_1 + \cdots + e_n$. Such a set of idempotents of A corresponds to decomposition of A into a product

$$A \cong A_1 \times \cdots \times A_n,$$

where $A_i = e_i \cdot A \subset A$ is a subring of A.

(a) Let $\mathfrak{a} \subset A$ be an ideal. Show that $\mathfrak{a} = \mathfrak{a}_1 + \cdots + \mathfrak{a}_n$, where $\mathfrak{a}_i = e_i \cdot \mathfrak{a}$.

(b) Show that in item (a), $\mathfrak{a}$ is a prime ideal if and only if there is an index i_0 such that $\mathfrak{a}_i = A_i$ for all $i \neq i_0$ and $\mathfrak{a}_{i_0} \subset A_{i_0}$ is prime.

(c) Show that $\operatorname{Spec} A_i \subset \operatorname{Spec} A$ is open and closed for all i, and that

$$\operatorname{Spec} A = \operatorname{Spec} A_1 \sqcup \cdots \sqcup \operatorname{Spec} A_n$$

is the disjoint union of the $\operatorname{Spec} A_i$. In particular, $\operatorname{Spec} A$ is not connected if $n \geq 2$.

Exercise 3. (4 points) *When is* $\operatorname{Spec} A$ *connected?*
Let A be a ring. Show that $\operatorname{Spec} A$ is connected if and only if the only idempotent elements of A (i.e. elements $a \in A$ with $a^2 = a$) are 0 and 1.

Exercise 4. (4 points) *Gluing two copies of* $\mathbb{A}^1_k$
Let k be a field and let

$$X_1 := \operatorname{Spec} k[t] \quad \text{and} \quad X_2 := \operatorname{Spec} k[s]$$

be two copies of the affine line $\mathbb{A}^1_k$ with open subsets

$$U_1 := X_1 \setminus V(t) \cong \operatorname{Spec} k[t, t^{-1}] \quad \text{and} \quad U_2 := X_2 \setminus V(s) \cong \operatorname{Spec} k[s, s^{-1}].$$

Consider the scheme X that one obtains by gluing X_1 and X_2 along the open subsets U_1 and U_2 via the isomorphism $\tau : U_1 \xrightarrow{\sim} U_2$, induced by one of the isomorphisms of rings

(a) $k[t, t^{-1}] \to k[s, s^{-1}], \quad t \mapsto s,$

(b) $k[t, t^{-1}] \to k[s, s^{-1}], \quad t \mapsto s^{-1}.$

In each of the cases (a) and (b) above, describe the topological space X (draw a picture!) and compute $\Gamma(X, \mathcal{O}_X) := \mathcal{O}_X(X)$. Decide also in each case whether X is isomorphic to an affine scheme.

Hand in: before noon on Monday, May 20th in the appropriate box on the 1st floor.