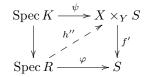
Exercise 4:

(c) Proper morphisms are stable under base change.

Let $f: X \to Y$ be proper and $g: S \to Y$ be any morphism. We want to show that the induced morphism $f': X \times_Y S \to S$ is proper. Notice that f' is of finite type by previous excercise. Now for any commutative diagram

$$\operatorname{Spec} K \xrightarrow{\psi} X \times_Y S \xrightarrow{g'} X$$
$$\downarrow \overset{h'}{\xrightarrow{\varphi}} \overset{\varphi'}{\xrightarrow{\varphi}} \overset{\varphi'}{\xrightarrow{\varphi}} X$$
$$\downarrow g \xrightarrow{f'} \overset{\varphi'}{\xrightarrow{\varphi}} Y$$

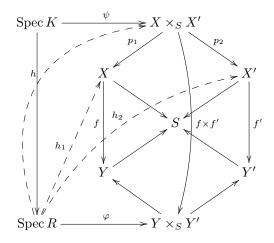
where R is a valuation ring of field K. Then since f is proper, there exists a unique morphism $h : \operatorname{Spec} R \to X$ making the above diagram commutative. Now by universal property of fiber product $X \times_Y S$, we have a unique morphism $h' : \operatorname{Spec} R \to X \times_Y S$ making the above diagram commutative. Note that h' is unique. In fact, if there is another morphism h'' making the following diagram commutative



then since $g' \circ h''$ is another lifting of $g \circ \varphi$, we have $g' \circ h'' = h = g' \circ h'$, by valuation criterion for f. Then h' = h'' by universal property of fiber product.

(d) Product of proper morphisms is a proper morphism.

Let $f: X \to Y$ and $f': X' \to Y'$ be proper over S, by unversal property we have $f \times f': X \times_S X' \to Y \times_S Y'$. We want to show that $f \times f'$ is proper. Note that $f \times f'$ is locally of finite type, which can be checked locally. Also, since $X \times_S X'$ is notherian, we have $f \times f'$ is of finite type. Now for any commutative diagram



by properness of f and f', there are unique morphisms h_1 : Spec $R \to X$ and h_2 : Spec $R \to X'$, repectively, making the corresponding diagrams commutative. Then by universal property of fiber product $X \times_S X'$, we get an unique morphism h: Spec $R \to X \times_S X'$ making the above diagram commutative. Note that h satisfy the uniqueness for the valuation criterion for $f \times f'$. In fact, if h' is another lifting of φ , then by valuation criterion for f and f', $h_1 = p_1 \circ h$ and $h_2 = p_2 \circ h$. Then by universal property of fibre product $X \times_S X'$, we get h = h'.

(e) Please refer to Hartshorne algebraic geometry, page 102 proof of corollary 4.8.

(f) Properness is local on the base.

We want to show that $f: X \to Y$ is proper if only if there exists an open cover $Y = \bigcup_i V_i$ such that the induced map $f_i: f^{-1}(V_i) \to V_i$ is proper for all *i*. Note first, *f* and f_i are of finite type.

" \Leftarrow " For any commutative diagram

Take a open set V_i such that it contains the image of the closed point $(0) \in R$ under φ . Then the above diagram factors through $f^{-1}(V_i) \to V_i$

By properness of f_i , there exists a unique morphism $h_i : \operatorname{Spec} R \to f^{-1}(V_i)$ making the diagram commutative. Then $j \circ h_i$ is a lifting of $\varphi(=i \circ \varphi)$. It is unique since another lifting h will factor through j and $j : f^{-1}(V_i) \to X$ is an open immersion. Then the unqueness of $j \circ h_i$ coming from the properness of f_i . " \Rightarrow " For any commutative diagram

we have the extended commutative diagram

since f is proper, we have a unique lifting h of $i \circ \varphi_i$. Since i and j are open immersion, we have h factors through $j: f^{-1}(V_i) \to X$, i.e., there exists a morphism $h_i: \operatorname{Spec} R \to f^{-1}(V_i)$ such that $h = j \circ h_i$. h_i is a lifting of φ_i . The uniquenes of h_i follows immediately from the uniquess of h.