Exercise 4:
(c) Proper morphisms are stable under base change.
Let $f : X \to Y$ be proper and $g : S \to Y$ be any morphism. We want to show that the induced morphism $f' : X \times_Y S \to S$ is proper. Notice that f' is of finite type by previous excercise. Now for any commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "\psi"] \arrow[d] & X \times_Y S \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r, "\varphi"] \arrow[ur, dashed, "h'"] \arrow[urr, dashed, "h"] & S \arrow[r, "g"] & Y
\end{tikzcd}$$

where R is a valuation ring of field K. Then since f is proper, there exists a unique morphism $h : \operatorname{Spec} R \to X$ making the above diagram commutative. Now by universal property of fiber product $X \times_Y S$, we have a unique morphism $h' : \operatorname{Spec} R \to X \times_Y S$ making the above diagram commutative. Note that h' is unique. In fact, if there is another morphism h'' making the following diagram commutative

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "\psi"] \arrow[d] & X \times_Y S \arrow[d, "f'"] \\
\operatorname{Spec} R \arrow[r, "\varphi"] \arrow[ur, dashed, "h''"] & S
\end{tikzcd}$$

then since $g' \circ h''$ is another lifting of $g \circ \varphi$, we have $g' \circ h'' = h = g' \circ h'$, by valuation criterion for f. Then $h' = h''$ by universal property of fiber product.

(d) Product of proper morphisms is a proper morphism.
Let $f : X \to Y$ and $f' : X' \to Y'$ be proper over S, by unversal property we have $f \times f' : X \times_S X' \to Y \times_S Y'$. We want to show that $f \times f'$ is proper. Note that $f \times f'$ is locally of finite type, which can be checked locally. Also, since $X \times_S X'$ is notherian, we have $f \times f'$ is of fintie type. Now for any commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[rrr, "\psi"] \arrow[dddd] & & & X \times_S X' \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dddd, bend left, "f \times f'"] & \\
& & X \arrow[dr] \arrow[dd, "f"] & & X' \arrow[dl] \arrow[dd, "f'"] \\
& & & S & \\
& & Y \arrow[ur] & & Y' \arrow[ul] \\
\operatorname{Spec} R \arrow[rrr, "\varphi"] \arrow[uuuurrr, dashed, bend left, "h"] \arrow[uuurr, dashed, "h_1"] \arrow[uuurrrr, dashed, "h_2"] & & & Y \times_S Y' \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

by properness of f and f', there are unique morphisms $h_1 : \operatorname{Spec} R \to X$ and $h_2 : \operatorname{Spec} R \to X'$, repectively, making the corresponding diagrams commutative. Then by universal property of fiber product $X \times_S X'$, we get an unique morphism $h : \operatorname{Spec} R \to X \times_S X'$ making the above diagram commutative. Note that h satisfy the uniqueness for the valuation criterion for $f \times f'$. In fact, if h' is another lifting of φ, then by valuation criterion for f and f', $h_1 = p_1 \circ h$ and $h_2 = p_2 \circ h$. Then by universal property of fibre product $X \times_S X'$, we get $h = h'$.

(e) Please refer to Hartshorne algebraic geometry, page 102 proof of corollary 4.8.

(f) Properness is local on the base.
We want to show that $f : X \to Y$ is proper if only if there exists an open cover $Y = \bigcup_i V_i$ such that the induced map $f_i : f^{-1}(V_i) \to V_i$ is proper for all i. Note first, f and f_i are of fintie type.

"⇐" For any commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "\psi"] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r, "\varphi"] & Y
\end{tikzcd}$$

Take a open set V_i such that it contains the image of the closed point $(0) \in R$ under φ. Then the above diagram factors through $f^{-1}(V_i) \to V_i$

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "\psi"] \arrow[d] & f^{-1}(V_i) \arrow[r, hook, "j"] \arrow[d, "f_i"] & X \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r, "\varphi"] \arrow[ur, dashed, "h_i"] & V_i \arrow[r, hook, "i"] & Y.
\end{tikzcd}$$

By properness of f_i, there exists a unique morphism $h_i : \operatorname{Spec} R \to f^{-1}(V_i)$ making the diagram commutative. Then $j \circ h_i$ is a lifting of $\varphi(= i \circ \varphi)$. It is unique since another lifting h will factor through j and $j : f^{-1}(V_i) \to X$ is an open immersion. Then the unqiueness of $j \circ h_i$ coming from the properness of f_i.

"⇒" For any commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "\psi_i"] \arrow[d] & f^{-1}(V_i) \arrow[d, "f_i"] \\
\operatorname{Spec} R \arrow[r, "\varphi_i"] & V_i
\end{tikzcd}$$

we have the extended commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "\psi_i"] \arrow[d] & f^{-1}(V_i) \arrow[r, hook, "j"] \arrow[d, "f_i"] & X \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r, "\varphi_i"] \arrow[urr, dashed, "h"] & V_i \arrow[r, hook, "i"] & Y.
\end{tikzcd}$$

since f is proper, we have a unique lifting h of $i \circ \varphi_i$. Since i and j are open immersion, we have h factors through $j : f^{-1}(V_i) \to X$, i.e., there exists a morphism $h_i : \operatorname{Spec} R \to f^{-1}(V_i)$ such that $h = j \circ h_i$. h_i is a lifting of φ_i.The uniquenes of h_i follows immediately from the uniquess of h.