Exercise 1:b)

If $d > 1$. At the point $p = (x_0, \ldots, x_n) \in \operatorname{Proj} S$, we have $\mathcal{O}_X(1)_p = \widetilde{S(1)}_p = S(1)_{(p)}$. Assume by contradiction that $\mathcal{O}_X(1)$ is locally free by the proof of Ex1 (a), it is of rank 1. Then this means especially there exists an isomorphism

$$\varphi : S_{(p)} \to S(1)_{(p)}; 1 \mapsto \frac{s}{f},$$

with $\deg s = \deg f + 1$ in $k[y, x_0, \ldots, x_n]$ and $f \notin p$ is homogenus. We noticed also, since $f \notin p$ and homogeneous, there is a term ay^N of f, with $a \neq 0$. Thus $\deg f = dk$ with $k \in \mathbb{Z}^{\geq 0}$. Also, we can assume that $(s, f) = 1$ in polynomial ring $k[y, x_0, \ldots, x_n]$.

Now through the above isomorphism φ, we have for elements $\frac{x_i}{1} \in S(1)_{(p)}$, there exist $\frac{s_i}{f_i} \in S_{(p)}$ such that $\frac{ss_i}{ff_i} = \frac{x_i}{1}$ with $\deg s_i = \deg f_i$. Also, as for $\frac{s}{f}$, we have $\deg f_i = dk' = \deg s_i$ with $k' \in \mathbb{Z}^{\geq 0}$ and we can assume $(f_i, s_i) = 1$. Then for each i, we have

$$ss_i = ff_ix_i.$$

Since $(f, s) = 1$, we have $f|s_i$, similarly, $f_i|s$. Hence $\deg s \geq \deg f_i$ and $\deg s_i \geq \deg f$. Also, we have $\deg s = \deg f + 1$ and $\deg s_i = \deg f_i$. Hence $s_i = \alpha_i f$ or $s = \alpha_i f_i$ with $0 \neq \alpha_i \in k$. However, $d|\deg s_i$, $d|\deg f$, and $d > 1$. hence $s_i = \alpha_i f$ and $s = \beta_i f_i x_i$ for $0 \neq \alpha_i, \beta_i \in k$. Then for $i \neq j$, we get $\beta_i f_i x_i = \beta_j f_j x_j$, which implies $x_i|f_j$. This contradicts to the fact that $f_i \notin p$.

Ex4. The conterexample is: We take polynomial ring $S = k[x]$ for k a field. Then $X = \operatorname{Proj} k[x]$ is one point, which is not empty. Now take graded module $M = \frac{k[x]}{x^m}$, $m > 1$. Then the associated $\mathcal{O}_X$ modules $\widetilde{M} = 0$, since higher degree terms are 0 for M. Also, one can check the stalk of $\widetilde{M}$ at the only point (0) of X. We get $\widetilde{M}_{(0)} = k[x]_{((0))}$. Any element of $k[x]_{((0))}$ say $\frac{f}{x^n} = 0$ since $x^N f = 0$ for $\deg x^N f > m - 1$.