Bezout's theorem

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Let X be a smooth projective curve, i.e. a smooth projective variety of dimension one. A divisor D on X is nothing but a $\mathbb{Z}$-linear combination of points on X:

$$D = \sum_i a_i[x_i],$$

with $a_i \in \mathbb{Z}$ and $x_i \in X$. We define the degree of D as

$$\deg D = \sum a_i.$$

This yields a group homomorphism

$$\deg : \operatorname{Div} X \longrightarrow \mathbb{Z}.$$

We have seen in the lecture that this descends to a group homomorphism

$$\deg : \operatorname{Cl}(X) \longrightarrow \mathbb{Z},$$

i.e. $\deg(\operatorname{Div}(f)) = 0$ for all $f \in k(X)^*$.

We aim to apply this result to prove a version of Bezout's theorem. For this, let $F, G \in k[x_0, x_1, x_2]$ be non-constant irreducible (or more general, square-free, which means that in the decomposition of F and G into powers of irreducible factors, each irreducible factor appears with exponent one) homogeneous polynomials, and consider the corresponding plane curves

$$X := V(F) \subset \mathbb{P}^2 \quad \text{and} \quad Y := V(G) \subset \mathbb{P}^2.$$

We assume that X and Y have no component in common and aim to compute the number of intersection points $X \cap Y$, counted with the correct multiplicities. For simplicity, we assume that X is smooth (for the general case, one may pass to the normalization of X which is a smooth projective model of X). We then define $\sharp(X \cap Y)$ as follows. Let $E \in k[x_0, x_1, x_2]$ be a homogeneous polynomial of degree $\deg G$, such that

$$V_{\mathbb{P}^2}(F, G, E) = \emptyset.$$

Then

$$f := \frac{G}{E}|_X \in k(X)$$

is a rational function whose divisor of zeros and poles

$$\operatorname{Div}(f) = D - D'$$

with $D, D' \geq 0$ and such that D and D' have no point in common, has the property that D does not depend on E. We may then define

$$\sharp(X \cap Y) = \deg D.$$

Theorem 0.1 (Bezout's Theorem)**.** *In the above notation*

$$\sharp(X \cap Y) = \deg F \cdot \deg G.$$

Proof. Let $L \in k[x_0, x_1, x_2]$ be a linear homogeneous polynomial such that $V(L)$ is not tangent to X at any $x \in X$. (This is possible, because the lines in $\mathbb{P}^2$ are parametrized by $\mathbb{P}(k[x_0, x_1, x_2]_{(1)}) \cong \mathbb{P}^2$, while the lines that are tangent to X correspond to the image of the regular map

$$X \to \mathbb{P}(k[x_0, x_1, x_2]_{(1)}), \quad x \mapsto d_x F$$

and so they form a subset of dimension at most one of $\mathbb{P}(k[x_0, x_1, x_2]_{(1)}) \cong \mathbb{P}^2$.) We may additionally assume that

$$V(F, G, L) = \emptyset.$$

Hence, in the definition of $\sharp(X \cap Y)$ we can take $E = L^{\deg G}$. Then,

$$f := \frac{G}{E}|_X \in k(X)$$

is a rational function on X and we can write

$$\operatorname{Div}(f) = D - D'$$

where $D, D' \geq 0$ are effective and have no points in common. Since $\deg(\operatorname{Div}(f)) = 0$, we find that

$$\sharp(X \cap Y) = \deg D = \deg D'.$$

Since L is not tangent to X at any point, we have for all $x \in X \cap V(L)$ that the linear map

$$d_x L : T_{X,x} \to k$$

is surjective. For $x \in X \cap V(L)$, we conclude that the image of L in $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ is a generator and so L generates the maximal ideal $\mathfrak{m}_{X,x} \subset \mathcal{O}_{X,x}$ by Nakayama's lemma. Since $V(F, G, L) = \emptyset$, G does not vanish at $x \in X \cap V(L)$ and so

$$\frac{G}{L^{\deg G}} \in \operatorname{Frac}(\mathcal{O}_{X,x}),$$

where $G \in \mathcal{O}_{X,x}$ is a unit and $L \in \mathcal{O}_{X,x}$ is a uniformizer. Hence, $x \in X \cap V(L)$ appears in $\operatorname{Div}(f)$ with coefficient $-\deg G$ and so we conclude

$$\sharp(X \cap Y) = \deg D' = \deg G \cdot \sharp(V(L) \cap X),$$

where $\sharp(V(L) \cap X)$ denotes the number of intersection points of $V(L)$ and X. Since for all $x \in V(L) \cap X$, the image of L in $\mathcal{O}_{X,x}$ is a uniformizer, we find that L vanishes of order one at x. Equivalently, the homogeneous polynomial F which cuts out X vanishes of order one at $x \in V(L)$. That is, the restriction of F to $V(L) \cong \mathbb{P}^1$ is a polynomial of degree $\deg F$ without multiple zeros and so it has exactly $\deg F$ many zeros. That is,

$$\sharp(V(L) \cap X) = \deg F$$

and so

$$\sharp(X \cap Y) = \deg F \cdot \deg G,$$

as we want. $\square$