## SHEET4, EX1

**Ex1:** After doing linear change of coordinates, we can assume $0 \in X$. Let $\lambda_i = \frac{x_i}{x_1}, i \geq 2$. After plugging $x_i = \lambda_i x_1$ into the defining polynomial $f$, we get

$$f(x_1, \lambda_2 x_1, \dots, \lambda_n x_1) = x_1(a(\lambda_2, \dots, \lambda_n)x_1 - b(\lambda_2, \dots, \lambda_n)),$$

with $a, b \in k[\lambda_2, \dots, \lambda_n]$.

Case 1: If $b \neq 0$, then we use the pair of rational maps

$$\varphi : X \dashrightarrow \mathbb{A}_k^{n-1}; (x_1, \dots, x_n) \mapsto (\frac{x_2}{x_1}, \dots, \frac{x_n}{x_1})$$

(here WLOG we assume there is a point $(a_1, \dots, a_n)$ in X such that $a_1 \neq 0$), and

$$\psi : \mathbb{A}_k^{n-1} \dashrightarrow X; (\lambda_2, \dots, \lambda_n) \mapsto (\frac{b}{a}, \frac{b}{a}\lambda_2, \dots, \frac{b}{a}\lambda_n),$$

where $a, b$ are polynomials defined above.

Case 2: If $b = 0$, then $f$ is of form $a_1 x_1^2 + \dots + a_n x_n^2$ ($n \geq 3$ and at least three of $a_i$ are nonzero, by irreducibility of $f$) after a linear change of coordinates. Then we need to choose another starting point (in case 1, it is the origin) to project $X$ to affine space $\mathbb{A}_k^{n-1}$. With out losing of generality, we can assume that $a_1 \neq 0, a_2 \neq 0$ and $a_3 \neq 0$. For this we need to do the following change of coordinates:

$$y_1 = x_1 - 1, y_2 = x_2 - \xi, y_i = x_i \text{ for } i \neq 1, 2,$$

where $\xi \in k$ and $\xi^2 = -\frac{a_1}{a_2}$. Then after doing this change of coordinates, we get

$$f = a_1 y_1^2 + a_2 y_2^2 + \dots + a_n y_n^2 + 2a_1 y_1 + 2\xi a_2 y_2.$$

Since $a_1, a_2, a_3$ are nonzero, there are two of them such that $a_i \neq -a_j$, say $a_1 \neq -a_2$. Then $\xi \neq 1$. Hence plugging in $y_i = \lambda_i y_1$, we get

$$f = y_1(a(\lambda_2, \dots, \lambda_n)y_1 - b(\lambda_2, \dots, \lambda_n)),$$

with $b \neq 0$. Then as that in case 1. We define the pair of rational maps

$$\varphi : X \dashrightarrow \mathbb{A}_k^{n-1}; (y_1, \dots, y_n) \mapsto (\frac{y_2}{y_1}, \dots, \frac{y_n}{y_1})$$

and

$$\psi : \mathbb{A}_k^{n-1} \dashrightarrow X; (\lambda_2, \dots, \lambda_n) \mapsto (\frac{b}{a}, \frac{b}{a}\lambda_2, \dots, \frac{b}{a}\lambda_n),$$

where $a, b$ are polynomials defined above. Then it is easy to check that this pair of rational maps will show that $\varphi$ is a birational map.