SHEET 2

Exercise 1. (d)

1) *Claim that* $S = \{(t, 2^t) \mid t \in \mathbb{Z}\}$ *is dense.*

Proof: It suffices to show that $I(S) = (0)$.

Assume that $I(S) \neq (0)$. We take $f \in I(S)$ such that $f \neq 0$. Write $f(x, y) = g_m(x)y^m + g_{m-1}(x)y^{m-1} + \ldots + g_0(x)$, with $g_i(x)$ being polynomials and $g_m(x) \neq 0$. It is clear that $m > 0$. Plugging in the discrete points $(t, 2^t)$ into the polynomial f, we get

$$g_m(t)2^{tm} + g_{m-1}(t)2^{t(m-1)} + \ldots + g_0(t) = 0.$$

Pulling out 2^{tm}, which is positive, we get

$$g_m(t) + g_{m-1}(t)2^{-t} + \ldots + g_0(t)2^{-tm} = 0.$$

Since $\lim_{t\to\infty} g_m(t) = \infty$ or $-\infty$, and $\lim_{t\to\infty}(g_{m-1}(t)2^{-t} + \ldots + g_0(t)2^{-tm}) = 0$, we get a contradiction. Thus $I(S) = (0)$.

Exercise 4.

Proof: By the Hint of the problem, we consider

$$\psi : Y \to \mathbb{A}^n; y \to (\phi^{*-1}(\bar{t}_1)(y), \ldots, \phi^{*-1}(\bar{t}_n)(y)),$$

where $\bar{t}_i$ are image of the coordinate functions t_i in $k[X]$. We want to show that ψ is the inverse of ϕ.

1) First we show that $\psi(Y) \subset X$.

We just need to show that for any $g \in I(X) \subset k[t_1, \ldots, t_n]$, $g(\psi(y)) = 0$. First note that we have

$$g(\psi(y)) = g(\phi^{*-1}(\bar{t}_1)(y), \ldots, \phi^{*-1}(\bar{t}_n)(y)) = g(\phi^{*-1}(\bar{t}_1), \ldots, \phi^{*-1}(\bar{t}_n))(y),$$

with $g(\phi^{*-1}(\bar{t}_1), \ldots, \phi^{*-1}(\bar{t}_n))$ being a regular functions in $k[Y]$. Since $\phi^{*-1} : k[X] \to k[Y]$ is a ring isomorphism, we have

$$g(\phi^{*-1}(\bar{t}_1), \ldots, \phi^{*-1}(\bar{t}_n)) = \phi^{*-1}g(\bar{t}_1, \ldots, \bar{t}_n),$$

with $g(\bar{t}_1, \ldots, \bar{t}_n) \in k[X]$. Note also, $g(\bar{t}_1, \ldots, \bar{t}_n) = 0$ in $k[X]$, since $g \in I(X)$ and $g(t_1, \ldots, t_n)(x) = g(x_1 \ldots, x_n) = 0$ for all $x \in X$. Thus we get $g(\psi(y)) = 0$. Hence we get a regular map $\psi : Y \to X$.

2) We show that $\psi \circ \phi = id$.

For any $x \in X$, write $x = (x_1, \ldots, x_n) \in \mathbb{A}^n$. Then

$$\psi \circ \phi(x) = (\phi^{*-1}(\bar{t}_1), \ldots, \phi^{*-1}(\bar{t}_n))(\phi(x)).$$

For each $\phi^{*-1}(\bar{t}_i) \in k[Y]$, we have

$$\phi^{*-1}(\bar{t}_i)(\phi(x)) = \phi^*\phi^{*-1}(\bar{t}_i)(x) = x_i.$$

Thus $\psi \circ \phi(x) = x$.

3) We show that $\phi \circ \psi = id$.

Now assume that $Y \in \mathbb{A}^m$ for some m, and $s_1, \dots, s_m$ are coordinate functions of $\mathbb{A}^m$, with $\bar{s}_i$ are the images of s_i in $k[Y]$. Now we take $y \in Y$, and write $y = (y_1, \dots, y_m) \in \mathbb{A}^m$. Then

$$\phi \circ \psi(y) = (\phi^*(\bar{s}_1), \dots, \phi^*(\bar{s}_m))(\psi(y)).$$

For each $\phi^*(\bar{s}_i) \in k[X]$, we have

$$\phi^*(\bar{s}_i)(\psi(y)) = \psi^* \phi^*(\bar{s}_i)(y) = y_i.$$

Thus $\phi \circ \psi(y) = y$.