Ex 2.

Proof. It suffices to show if $f : X \dashrightarrow Y$ is a birational map, then $X \cong Y$.
Let $g : Y \dashrightarrow X$ be the inverse of f. Then the compositions $f \circ g$ and $g \circ f$ are identities on an open set $U \subset X$ and $V \subset Y$, resp. Also, by Ex 1, we know that f and g can be extended to regular maps. Hence $f \circ g : Y \to Y$ is a regular map which is an identity on an open set V of Y, so is $g \circ f$. Assume $Y \subset \mathbb{P}^n$ with homogeneous coordinates $x_0, x_1, \ldots, x_n$. For any points $x = [x_0 : \ldots : x_n] \in V$, $f \circ g(x) = [F_0(x) : \ldots : F_n(x)] = [x_0 : \ldots : x_n]$. Hence $f \circ g(x) = x$ for any $x \in Y$. $\square$

Ex 5. (d) Let $X = \mathbb{P}^1$ and $Y := V_{\mathbb{P}^2}(x_2x_1^2 - x_0^3)$. Let

$$\phi : X \to Y$$

be the morphism maps $[t, 1]$ to $[t^2, t^3, 1]$ and $[1 : 0]$ to $[0 : 1 : 0]$. Then the restriction map $\phi|_{X-[1:0]} : \mathbb{A}^1 \to V(X_1^2 - X_0^3) \subset \mathbb{P}^2 - V(x_2) \cong \mathbb{A}^2$, where $X_i = \frac{x_i}{x_2}$, is the normalization map of the cusp in Homework sheet 9, Ex3.

(e) For any integer $n \geq 1$, we take the following curve in $\mathbb{P}^2$ with homogeneous coordinates $[x : y : z]$.

$$V_{\mathbb{P}^2}(x(x - a_1z)^2 \ldots (x - a_nz)^2 + y^{2n+1}),$$

with a_i are distinct from each other and $a_i \neq 0$. Then using Jacobian criterion, we see that the only singular points are $[a_1 : 0 : 1], [a_2 : 0 : 1], \ldots, [a_n : 0 : 1]$.

(f) If X is an affine and projective variety. Notice first any regular function over X is a constant function. Then $K[X] \cong k$. Hence $X \cong pt$.

(g) X is a quasi-projective variety. If $x \in X$ is a singular point, then $\dim T_{X,x} > \dim_x X = \dim X$. If $x \in X$ is a smooth point, then $\dim T_{X,x} = \dim_x X = \dim X$. Also, since the smooth locus of X is not empty. Thus $\dim X = min\{\dim T_{X,x} \mid x \in X\}$.

(i) $X = V(F) \subset \mathbb{P}^n$. We can assume that the homogeneous polynomial $F = F_1 \ldots F_s$, with F_i being irreducible, and F_i and F_j are mutrually coprime. Assume X is not irreducible, then $s > 1$. Let $G = F_2 \ldots F_s$. Then we have $X = V(F_1) \cup V(G)$, and $V(F_1) \cap V(G) \neq \emptyset$ by Krull's Hauptidealsatz. Then by Homework 8 Ex4, we have for any $x \in V(F_1) \cap V(G)$, x is a singular point of X, which is a contradiction.

(j)
(1) Let $f_1 = x_0^2 - x_0x_2 - x_1x_3$ and $f_2 = x_1x_2 - x_0x_3 - x_2x_3$. Then the Jacobian Matrix of f_1, f_2 is

$$\begin{bmatrix} 2x_0 - x_2 & -x_3 & -x_0 & -x_1 \\ -x_3 & x_2 & x_1 - x_3 & -x_0 - x_2 \end{bmatrix}$$

Assume that there is a singular point $[x_0 : x_1 : x_2 : x_3]$ in X. Note first $x_2 \neq 0$, otherwise, $x_3 = 0$, which implies $x_i = 0$ for all i.
Let λ be a constant such that 1) $2x_0 - x_2 = -\lambda x_3$; 2) $-x_3 = \lambda x_2$; 3) $-x_0 = \lambda(x_1 - x_3)$; 4) $-x_1 = \lambda(-x_0 - x_2)$. 4)+3)+2) implies $(-\lambda^2 - 1)x_0 = 2\lambda^2 x_2$. Also, 1)+2) implies $2x_0(\lambda^2 + 1)x_2$. Hence we have $-(\lambda^2 + 1)^2 = 4\lambda^2$ by $x_2 \neq 0$. Hence $\lambda \neq 0$. Hence $x_3 \neq 0$.
Now by second and last column of the Jacobian matrix, we have $x_0x_3 + x_2x_3 + x_1x_2 = 0$. Thus $x_2 = 0$ by $f_2 = 0$. Hence $x_1 = 0$. Plug $x_1 = 0$ in $f_2 = 0$, we have $x_0 = -x_2$. Plug $x_0 = -x_2$ in $f_1 = 0$, we get $x_0 = x_2 = 0$, which is a contradiction. Hence X is smooth.

(2) Let $f_1 = x_0x_1 - x_2^2 - x_3^2$ and $f_2 = x_0x_1 + x_2x_3 + x_4^2$. Then the Jacobian Matrix of f_1, f_2 is

$$\begin{bmatrix} x_1 & x_0 & -2x_2 & -2x_3 & 0 \\ x_1 & x_0 & x_3 & x_2 & 2x_4 \end{bmatrix}$$

Assume that there is a singular point $[x_0 : x_1 : x_2 : x_3]$ in X. Let λ be a constant such that 1) $x_1 = \lambda x_1$; 2) $x_0 = \lambda x_0$; 3) $-2x_2 = \lambda x_3$; 4) $-2x_3 = \lambda x_2$; 5) $0 = \lambda 2x_4$.

First notice that $\lambda \neq 0$, otherwise $x_0 = x_1 = x_2 = x_3 = x_4 = 0$, which is a contradiction. Thus $\lambda \neq 0$, and $x_4 = 0$. Claim that $x_2 = 0$. If not, then x_1, x_2, x_3 are all nonzero. Hence $\lambda = 1$. Then we get $x_2 = 0$ by equations 3)4), which is a contradiction. Thus we get singular points $[1 : 0 : 0 : 0 : 0]$, $[0 : 1 : 0 : 0 : 0]$.