Winter term 2021

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Mathematical Gauge Theory II

Sheet 2

Exercise 1. (Anti-linear automorphism II) Let $J, J' \colon V \to V'$ be complex anti-linear isomorphisms between standard Clifford modules. Show that there exists a number $\lambda \in S^1$ so that $J'(\lambda\phi) = \lambda J(\phi)$ for all $\phi \in V$.

Exercise 2. (The group $\mathrm{Spin}(n)$) Let $J_0 \colon \mathbb{C}^N \to \mathbb{C}^N$ be a complex anti-linear automorphism of the standard Clifford module γ_0. The group $\mathrm{Spin}(n)$ is defined as the set of pairs $(\tau, \sigma) \in \mathrm{Spin}^c(n)$ so that σ commutes with J_0. Prove that the homomorphism

$$q \colon \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$$
$$(\tau, \sigma) \longmapsto \tau$$

is surjective with kernel $\{(I_n, \pm I_N)\} \cong \mathbb{Z}_2$.

Exercise 3. ($\mathrm{Spin}^c(n)$ reconstructed from $\mathrm{Spin}(n)$) We consider the quotient

$$(\mathrm{Spin}(n) \times S^1)/\mathbb{Z}_2,$$

where (τ, σ, λ) gets identified with $(\tau, -\sigma, -\lambda)$. Prove that the homomorphism

$$(\mathrm{Spin}(n) \times S^1)/\mathbb{Z}_2 \longrightarrow \mathrm{Spin}^c(n)$$
$$[\tau, \sigma, \lambda] \longmapsto (\tau, \lambda\sigma)$$

is an isomorphism.

Exercise 4. (Fundamental representation of $\mathrm{SU}(2)$) Show that there exists a fixed matrix $M \in \mathrm{SU}(2)$ such that

$$MAM^\dagger = \bar{A} \quad \forall A \in \mathrm{SU}(2).$$

You can hand in solutions in the lecture on Thursday, 4 November 2021.