Winter term 2023/24 30 January 2024

Topology I

Sheet 14

Exercise 1. Let $(C, d_C), (D, d_D) \in \mathrm{Ch}(R)$ be chain complexes and let $f\colon C \to D$ be a chain map. The *cone of* f is the chain complex $C(f) \in \mathrm{Ch}(R)$ with $C(f)_n = D_n \oplus C_{n-1}$ and differential $d\colon C(f)_n \to C(f)_{n-1}$ given by

$$D_n \oplus C_{n-1} \xrightarrow{\begin{pmatrix} d_D & f \\ 0 & -d_C \end{pmatrix}} D_{n-1} \oplus C_{n-2}.$$

(This means that if we write elements of $C(f)_n$ as vectors $(x, y)^T$ with $x \in D_n$ and $y \in C_{n-1}$, then the differential is computed by multiplying from the left with the matrix above.) Let $C[1] \in \mathrm{Ch}(R)$ be the chain complex with $C[1]_n = C_{n-1}$ and differential $d = -d_C$.

(a) Show that $C(f)$ is a chain complex and that the obvious inclusion $i\colon D \to C(f)$ is a chain map.

(b) Show that there is a short exact sequence of chain complexes

$$0 \to D \xrightarrow{i} C(f) \to C[1] \to 0\,.$$

Show that the boundary map of the associated long exact sequence in homology is induced by f.

(c) Show that f is a quasi-isomorphism if and only if $C(f)$ is an exact chain complex.

(d) Show that f is a chain homotopy equivalence if and only if $C(f)$ is contractible. [By "contractible" we mean that the unique map $C(f) \to 0$ is a chain homotopy equivalence.]

(e) Let $P, Q \in \mathrm{Ch}(R)_{\geq 0}$ be non-negatively graded chain complexes of projective R-modules and let $f\colon P \to Q$ be a quasi-isomorphism. Show that f is a chain homotopy equivalence.

Exercise 2. Let G be a group and let R be a ring. The *group ring* RG is the ring obtained by taking the free R-module RG on the set G and equipping it with a product as follows: Any element of RG can be viewed as a finite "formal R-linear combination of elements of G" $\sum_{g\in G} r_g g$ with $r_g \in R$ and $r_g = 0$ for all but finitely many $g \in G$. The product on RG is defined by

$$\left(\sum_{g\in G} r_g g\right)\left(\sum_{g'\in G} r_{g'} g'\right) = \sum_{g,g'\in G} (r_g r_{g'})(gg')$$

using multiplication in R and in G. There is a ring homomorphism $\varepsilon\colon RG \to R$ defined by

$$\varepsilon\Big(\sum_{g\in G} r_g g\Big) = \sum_{g\in G} r_g$$

which makes R into an RG-module. Below RG will be viewed as an RG-module in this way.

Now suppose that G acts covering-like on a space X.

(a) Show that $C^{\mathrm{sing}}(X)$ is a chain complex of free $\mathbb{Z}G$-modules. Furthermore, show that there is an isomorphism of chain complexes

$$C^{\mathrm{sing}}(X) \otimes_{\mathbb{Z}G} \mathbb{Z} \cong C^{\mathrm{sing}}(X/G)\,.$$

(b) Suppose that G is finite with $|G| \in R^\times$. Show that R is a projective RG-module. [Hint: Consider the map $RG \to RG$ given by multiplication with $\frac{1}{|G|}\sum_{g\in G} g$.]

(c) Let M be a flat left R-module (for example, $M = R$). Show that

$$H_*(X/G; M) \cong H_*(X; R) \otimes_{RG} M$$

where on the right M is viewed as an RG-module via $\varepsilon\colon\ RG \to R$.

(d) Let p be a prime and let $L = L(p; q_1, \ldots, q_d)$ be a $(2d-1)$-dimensional lens space. Compute $H_*(L; \mathbb{F}_\ell)$ for any prime $\ell \neq p$.

(e) Suppose that C_2 acts covering-like on X. Show that there is a short exact sequence

$$0 \to C^{\mathrm{sing}}(X/C_2; \mathbb{F}_2) \to C^{\mathrm{sing}}(X; \mathbb{F}_2) \to C^{\mathrm{sing}}(X/C_2; \mathbb{F}_2) \to 0\,.$$

Using this sequence compute $H_*(\mathbb{R}P^n; \mathbb{F}_2)$ for all $n \geq 0$.

(f) Prove the Borsuk-Ulam theorem in any dimension by showing that there cannot exist a continuous map $S^n \to S^{n-1}$ which is equivariant for the antipodal action of C_2. [Hint: Use naturality of the sequence in (e).]

This sheet will be discussed in the week of 5 February 2024.