Winter term 2023/24 — 23 January 2024

# Topology I

Sheet 13

**Exercise 1.** Prove Corollary 4.35 from the lecture: Let $(X, x)$ and $(Y, y)$ be pointed spaces such that the respective basepoints have pointed contractible open neighbourhoods. Then, for all $n \geq 0$, there is a canonical isomorphism $\tilde{H}_n(X) \oplus \tilde{H}_n(Y) \cong \tilde{H}_n(X \vee Y)$.

**Exercise 2.** Let $R$ be a ring. Let there be given a commutative diagram of (left) $R$-modules

$$\begin{array}{ccccccccc} & & M' & \xrightarrow{i} & M & \xrightarrow{j} & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & N' & \xrightarrow{k} & N & \xrightarrow{l} & N'' & & \end{array}$$

of which both rows are exact sequences. Show that there is an exact sequence

$$\ker(f') \to \ker(f) \to \ker(f'') \to \operatorname{coker}(f') \to \operatorname{coker}(f) \to \operatorname{coker}(f'')\,.$$

**Exercise 3.** Let there be given a commutative diagram of (left) $R$-modules

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

of which both rows are exact. Prove:

(a) If $b$ and $d$ are surjective and $e$ is injective, then $c$ is surjective.

(b) If $b$ and $d$ are injective and $a$ is surjective, then $c$ is injective.

(c) If $b$ and $d$ are isomorphims, $a$ is surjective and $e$ is injective, then $c$ is an isomorphism.

(please turn)

**Exercise 4.** Let $R$ be a ring and let $I$ be the chain complex of $R$-modules with $I_0 = R \oplus R$, $I_1 = R$ and $I_n = 0$ for all $n \neq 0, 1$ and differential $d_I \colon I_1 \to I_0$ given by $d_I(r) = (r, -r)$. Let $C, D \in \mathrm{Ch}(R)$ be two chain complexes of left $R$-modules.

(a) Show that the total complex of $I \otimes_R C$ (as defined in Remark 4.50) takes the following form: In degree $n$ it is given by $(I \otimes_R C)_n = C_n \oplus C_n \oplus C_{n-1}$, and the differential is given by

$$\begin{aligned} d_C \colon C_n \oplus C_n \oplus C_{n-1} &\to C_{n-1} \oplus C_{n-1} \oplus C_{n-2} \\ (x, y, z) &\mapsto (dx + z, dy - z, -dz)\,. \end{aligned}$$

(b) Show that chain maps $I \otimes_R C \to D$ correspond one-to-one to triples $(f, g, h)$, where $f, g \colon C \to D$ are chain maps and $h$ is a chain homotopy between $f$ and $g$.

This sheet will be discussed in the week of 29 January 2024.