LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

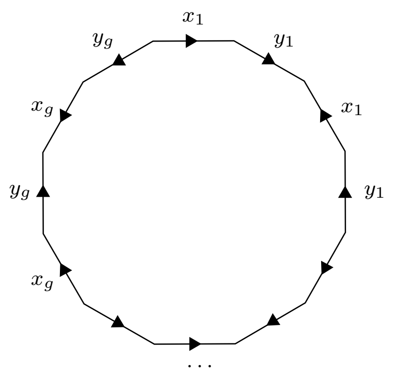
Winter term 2023/24 5 December 2023

Topology I

Sheet 8

Exercise 1. Consider the set $S = \{(\frac{2k+1}{2}, \frac{1}{2}) \mid k = 0, \ldots, n-1\} \subseteq \mathbb{R}^2$. Show that $\mathbb{R}^2 \backslash S$ is homotopy equivalent to a wedge $\bigvee^n S^1$. (If $S' \subseteq \mathbb{R}^2$ is more generally a subset of cardinality n, there is a homeomorphism $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi(S') = S$, hence also a homotopy equivalence $\mathbb{R}^2 \backslash S' \simeq \bigvee^n S^1$.)

Exercise 2. For an integer $g \geq 0$ let Σ_g be the quotient space obtained by identifying the sides of a filled regular $4g$-gon in $\mathbb{R}^2$ as in the following picture:



Show that there is a pushout diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{\alpha} & \bigvee^{2g} S^1 \\ \downarrow & & \downarrow \\ D^2 & \longrightarrow & \Sigma_g \end{array}$$

where $\alpha\colon S^1 \to \bigvee^{2g} S^1$ is an suitable map determined by the word $x_1 y_1 x_1^{-1} y_1^{-1} \cdots x_g y_g x_g^{-1} y_g^{-1}$.

(please turn)

Exercise 3. Let G be a group acting continuously on a space X. Recall that the action is *free* if, for all $x \in X$, $gx = x$ implies $g = e$. We call the action *proper* if, for all compact $K \subseteq X$, the set $\{g \in G \mid gK \cap K \neq \emptyset\}$ is finite.

(a) Show that if X is locally compact and Hausdorff, then the action of G on X is covering like if it is free and proper.

(b) Show that the implication of (a) does not hold if X is not assumed Hausdorff.

[Hint: For example, consider the space $X \sqcup X$ with topology $\{U \sqcup U \mid U \subseteq X \text{ open}\}$ and C_2-action given by swapping the two copies of X.]

Exercise 4. Let $p\colon E \to B$ be a covering map.

(a) Show that p is open.

(b) Show that if p is surjective, then p is a quotient map.

(c) Show that if B is connected and E is non-empty, then p is surjective.

(d) Show that if B is locally (path-)connected, then so is E. Show that this fails to hold if "locally" is replaced by "globally".

Exercise 5. Let $p\colon E \to B$ be a covering map and $f\colon B' \to B$ a map. Consider the pullback

$$\begin{array}{ccc} E \times_B B' & \longrightarrow & E \\ \Big\downarrow{\scriptstyle p'} & & \Big\downarrow{\scriptstyle p} \\ B' & \xrightarrow{\ f\ } & B \end{array}$$

(a) Show that the induced map $p'\colon E \times_B B' \to B'$ is also a covering map.

(b) Show that the assignment $(p, E, B) \mapsto (p', E \times_B B', B')$ induced by f defines a functor

$$f^*\colon \mathrm{Cov}(B) \to \mathrm{Cov}(B')$$

from the category of coverings over B to the category of coverings over B'.

This sheet will be discussed in the week of 11 December 2023.