Winter term 2023/24 28 November 2023

Topology I

Sheet 7

Exercise 1. Let there be given a commutative diagram of functors

$$\begin{array}{ccccc} \mathcal{C}_1 & \xrightarrow{F_1} & \mathcal{C}_0 & \xleftarrow{F_2} & \mathcal{C}_2 \\ \downarrow{\scriptstyle L_1} & & \downarrow{\scriptstyle L_0} & & \downarrow{\scriptstyle L_2} \\ \mathcal{D}_1 & \xrightarrow{G_1} & \mathcal{D}_0 & \xleftarrow{G_2} & \mathcal{D}_2 \end{array}$$

Show that it induces a canonical functor $\mathcal{C}_1 \hat{\times}_{\mathcal{C}_0} \mathcal{C}_2 \to \mathcal{D}_1 \hat{\times}_{\mathcal{D}_0} \mathcal{D}_2$. Furthermore show that this functor is

(1) fully faithful if L_i is fully faithful for all $i = 0, 1, 2$

(2) essentially surjective if L_1 and L_2 are essentially surjective and L_0 is fully faithful.

Conclude that it is an equivalence of categories if L_i is an equivalence of categories for all $i = 0, 1, 2$.

Exercise 2. Let $B\mathbb{Z}$ denote the infinite cyclic group $\mathbb{Z}$ viewed as a category with one object. Show that there is a natural equivalence of groupoids $\mathrm{Fun}(B\mathbb{Z}, \mathcal{G}) \simeq \mathcal{G} \hat{\times}_{\mathcal{G}\times\mathcal{G}} \mathcal{G}$, where the weak pullback is formed with respect to the diagonal inclusions $\mathcal{G} \to \mathcal{G} \times \mathcal{G}$.

Exercise 3. Let $F\colon \mathcal{G}_0 \to \mathcal{G}_1$ be a functor of small groupoids which is injective on objects, and let $\mathcal{G}$ be another groupoid. Show that the restriction functor $F^*\colon \mathrm{Fun}(\mathcal{G}_1, \mathcal{G}) \to \mathrm{Fun}(\mathcal{G}_0, \mathcal{G})$ is an isofibration.

Exercise 4. Let $C = \{(0, 0, z) \in \mathbb{R}^3 \mid -1 \leq z \leq 1\}$ and $X = S^2 \cup C$ viewed as a subspace of $\mathbb{R}^3$. Compute the fundamental group of X.

This sheet will be discussed in the week of 4 December 2023.