Winter term 2023/24 | 21 November 2023

Topology I

Sheet 6

Exercise 1. Consider the subspace $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}_{>0}\} \subseteq [0,1]$. Show that X is weakly equivalent to $\mathbb{N} \subseteq \mathbb{R}$, but not homotopy equivalent to it.

Exercise 2. Let $D^n \subseteq \mathbb{R}^n$ be the closed unit ball and let $i\colon S^{n-1} \to D^n$ be the inclusion of its boundary $(n-1)$-sphere. Let $\pi\colon S^{n-1} \to \mathbb{RP}^{n-1}$ be the quotient map, where $\mathbb{RP}^{n-1}$ is viewed as a quotient of S^{n-1} by the equivalence relation $x \sim -x$ for all $x \in S^{n-1}$.

a) Prove that there is a pushout square of the form

$$\begin{tikzcd} S^{n-1} \arrow[r, "\pi"] \arrow[d, "i"'] & \mathbb{RP}^{n-1} \arrow[d] \\ D^n \arrow[r] & \mathbb{RP}^n . \end{tikzcd}$$

Use this to define a CW-structure on $\mathbb{RP}^n$.

b) Similarly, construct CW-structures on $\mathbb{CP}^n$ and $\mathbb{HP}^n$.

Exercise 3. Classify all one-dimensional CW-complexes X up to homotopy equivalence.

[Hint: Assuming wlog that X is connected, you can find a maximal spanning tree $T \subseteq X$. Show that T is contractible and that $X/T \simeq \bigvee_I S^1$ is a wedge of circles.]

Exercise 4. Recall that an inclusion $i\colon A \to X$ is a cofibration if $X \times \{0\} \cup A \times [0,1]$ is a retract of $X \times [0,1]$.

a) Show that the inclusion $i\colon S^{n-1} \hookrightarrow D^n$ is a cofibration.

b) Show that the composition of cofibrations is a cofibration.

c) Show that for all X, Y the canonical map $X \to X \amalg Y$ is a cofibration.

d) Show that for all X the map $X \amalg X \to X \times [0,1]$ including the two ends of the cylinder is a cofibration.

e) Show that if $i\colon A \to X$ is a cofibration and Z is locally compact, then $i \times id\colon A \times Z \to X \times Z$ is a cofibration.

This sheet will be discussed in the week of 27 November 2023.