Winter term 2023/24 31 October 2023

Topology I

Sheet 3

Exercise 1. For a space X let $\pi_0(X)$ denote the set of path-connected components of X and let $\pi(X)$ denote the set of connected components of X.

(a) Show that the assignments $X \mapsto \pi_0(X)$ and $X \mapsto \pi(X)$ define functors Top $\to$ Set.

(b) Prove that π_0 and π commute with finite products.

Exercise 2. For a space X let $\pi(X)$ be the set of connected components endowed with the quotient topology induced by the canonical surjection $q\colon X \to \pi(X)$. Let td–spaces $\subseteq$ Top denote the full subcategory of totally disconnected spaces.

(a) Show that $X \mapsto \pi(X)$ defines a functor $\pi\colon$ Top $\to$ td–spaces; in particular, $\pi(X)$ with the quotient topology is totally disconnected.

(b) Show that $\pi\colon$ Top $\to$ td–spaces is left-adjoint to the inclusion td–spaces $\subseteq$ Top, that is, show that for every map $f\colon X \to Y$ where Y is totally disconnected there exists a unique map $\bar{f}\colon \pi(X) \to Y$ such that $\bar{f}q = f$.

(c) Let $\{X_i\}_{i\in I}$ be a family of totally disconnected spaces and $Y \subseteq \prod_{i\in I} X_i$ a subspace. Show that Y is totally disconnected. [Hint: Show that products of totally disconnected spaces are totally disconnected, and subspaces of totally disconnected spaces are totally disconnected.]

Exercise 3. Show that a space X is compact if and only if it satisfies the following condition: For every family $\{Z_i\}_{i\in I}$ of closed subsets of X, where I is a set, if $\bigcap_{j\in J} Z_j \neq \emptyset$ for all finite subsets $J \subset I$, then $\bigcap_{i\in I} Z_i \neq \emptyset$.

Exercise 4. Give an example of a space that is compact but not locally compact, and prove that this is so.

(please turn)

Exercise 5. For a map $p\colon X \to Y$ let $X \times_Y X$ denote the pullback of the diagram $X \xrightarrow{p} Y \xleftarrow{p} X$, that is, $X \times_Y X$ may be taken as the subspace of $X \times X$ defined by

$$X \times_Y X = \{(x_1, x_2) \in X \times X \mid p(x_1) = p(x_2)\}\,.$$

Let $\Delta = \{(x, x) \in X \times_Y X \mid x \in X\}$ be the diagonal subspace. Show that Δ is closed in $X \times_Y X$ if and only if for every $(x_1, x_2) \in X \times_Y X$ with $x_1 \neq x_2$ there are open sets $U_1, U_2 \subseteq X$ such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. In particular, deduce that X is Hausdorff if and only if the diagonal $\Delta \subseteq X \times X$ is closed.

This sheet will be discussed in the week of 6 November 2023.