TOPOLOGY I

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Abstract. These are lecture notes for my lecture "Topology I" which I taught in the winter term 2023/24 at LMU Munich.

Contents

The purpose of this lecture is to give an introduction to the field of algebraic topology, that is, the study of *topological spaces* by means of algebraic tools such as *homotopy groups* and *homology groups*. Of course, in order to do so, we must first define and study basic properties of topological spaces – this will be our first section on point-set topology. Next, we will define homotopy groups, and in particular the fundamental group and derive some basic properties

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including the theorem of Seifert and van Kampen. We will then discuss covering spaces and describe the intricate relationship between coverings of a given space X, a purely topological notion, and sets equipped with an action of the fundamental group of X. Finally, we will introduce the notion of a homology theory on topological spaces and construct the first non-trivial example: Singular homology. We will derive some standard applications, prove all relevant properties of singular homology (also known as the Eilenberg–Steenrod axioms), discuss the homology of CW complexes (i.e. that singular and cellular homology agree) and finish with some applications of the Euler characteristic.

This course will be followed by a course "Topology II" in the summer term 2024. There, we will discuss singular cohomology, its multiplicative structure and prove Künneth and universal coefficient theorems. We will move on and discuss the (co)homology of smooth/topological manifolds and prove Poincaré duality. If time permits, we will also discuss vector bundles and Thom isomorphisms. After this, we will go back to homotopy theory: We will discuss the notion of fibrations and associated long exact sequences in homotopy groups, simplicial sets, the relation between topological spaces and simplicial sets and in particular the theorem that a weak equivalence induces an isomorphism on singular homology, and representabilty of singular cohomology.

1. Point-set topology

We begin with the definition of a topological space. For a set X, we denote by $\mathcal{P}(X)$ its power set, i.e.

$$\mathcal{P}(X) = \{A \mid A \subseteq X\}.$$

1.1. Basic definitions.

1.1. **Definition** Let X be a set. A topology on X consists of a set $\mathrm{O} \subseteq \mathcal{P}(X)$, whose elements are called *open sets*, satisfying the following axioms:

(1) $X, \varnothing \in \mathrm{O}$,
(2) If $U_1, \dots, U_n \in \mathrm{O}$, then $U_1 \cap \cdots \cap U_n \in \mathrm{O}$,
(3) If I is a set and $U_i \in \mathrm{O}$ for all $i \in I$, then $\bigcup\limits_{i \in I} U_i \in \mathrm{O}$.

A pair (X, O) consisting of a set X and a topology O on X is called a *topological space.*

We begin with trivial examples.

1.2. **Example** Let X be a set. Then

(1) $\{\emptyset, X\}$ is a topology on X. It is called the *indiscrete topology.*
(2) $\mathcal{P}(X)$ is a topology on X. It is called the *discrete topology.*

We refer to the following basic (but useful) observation as the self-indexing trick:

1.3. **Observation** Let (X, O) be a topological space. Then $U \subseteq X$ is open if and only if for all $x \in U$, there exists $U_x \in \mathrm{O}$ such that $x \in U_x \subseteq U$.

Proof. If U is open and $x \in U$, we mau simply choose $U_x = U$. Conversely, note that $U = \bigcup\limits_{x \in U} U_x$, so U is open. □

1.4. **Notation** We will use the following terminology. Let (X, O) be a topological space.

(1) A subset $A \subseteq X$ is called *closed* if its complement $X \setminus A$ is open. We write $\mathcal{C}$ for the collection of closed subsets of $(X, \mathcal{O})$.
(2) For a point x in X, a subset $N \subseteq X$ containing x is called a *neighborhood of* x, if there exists an open set U such that $x \in U \subseteq N$.
(3) A subcollection $\mathcal{B}$ of a topology $\mathcal{O}$ is called a *basis for* $\mathcal{O}$ if every element in $\mathcal{O}$ is a union of elements in $\mathcal{B}$.
(4) A subcollection $\mathcal{S}$ of a topology $\mathcal{O}$ is called a *subbasis for* $\mathcal{O}$, if every element in $\mathcal{O}$ is a union of finite intersections of elements in $\mathcal{S}$.

1.5. **Remark** Let $(X, \mathcal{O})$ be a topological space. Then $\mathcal{O}$ is determined by the set of closed subsets of X: Indeed, the elements of $\mathcal{O}$ are precisely the complements in X of closed sets in X. One can therefore also describe a topology by specifying a subset $\mathcal{C} \subseteq \mathcal{P}(X)$ of closed sets of X satisfying the axioms

(1) $\varnothing, X \in \mathcal{C}$,
(2) If $A_1, \ldots, A_n \in \mathcal{C}$, then $A_1 \cup \cdots \cup A_n \in \mathcal{C}$,
(3) If I is a set, and $A_i \in \mathcal{C}$ for all $i \in I$, then $\bigcap_{i \in I} A_i \in \mathcal{C}$.

1.6. **Example** Let X be a set. Set $\mathcal{C} = \{A \subseteq X \mid A \text{ finite or equal to } X\}$ defines the closed sets of a topology on X, called the *cofinite topology.*

Next, we explain how to obtain many more examples of topologies.

1.7. **Lemma** *Let X and I be sets and for every $i \in I$, let $\mathcal{O}_i$ be a topology on X. Then the intersection $\mathcal{O} = \bigcap\limits_{i \in I} \mathcal{O}_i$ is again a topology on X.*

Proof. If I is empty, then $\mathcal{O} = \mathcal{P}(X)$ is indeed a topology. So let us assume that I is not empty. By assumption, $\emptyset, X \in \mathcal{O}_i$ for all $i \in I$. Hence, $\emptyset, X \in \mathcal{O}$. Likewise, if for $j = 1, \ldots, n$ we have $U_j \in \mathcal{O}$, then $U_j \in \mathcal{O}_i$ for all $i \in I$. Consequently, $U_1 \cap \cdots \cap U_n \in \mathcal{O}_i$ for all $i \in I$, and hence $U_1 \cap \cdots \cap U_n \in \mathcal{O}$ as needed. The same argument applies for arbitrary unions of elements of $\mathcal{O}$. □

1.8. **Definition** Let X be a set and $T \subseteq \mathcal{P}(X)$ a set of subsets of X. There exists a unique smallest topology $\mathcal{O}_T$ containing T. It is called the *topology generated by* T. The set T is a subbasis for the topology $\mathcal{O}_T$.

Proof. We define

$$\mathcal{O}_T = \bigcap \{\mathcal{O} \subseteq \mathcal{P}(X) \mid \mathcal{O} \text{ is a topology on X and } T \subseteq \mathcal{O}\}.$$

We note that this set over we take an intersection is non-empty since it contains $\mathcal{P}(X)$. $\mathcal{O}_T$ is a topology by Lemma 1.7. If $\mathcal{O}$ is any topology on X containing T, then by definition $\mathcal{O}_T \subseteq \mathcal{O}$, as $\mathcal{O}$ appears as one of the sets over which we form intersections. Hence $\mathcal{O}_T$ is the smallest topology which contains T. Now consider the set $\mathcal{S}_T$ consisting of arbitrary unions of finite intersections of elements of T. In formulas

$$\mathcal{S}_T = \{\bigcup_{i \in I} A_i \mid I \text{ is a set and } \forall i \in I\, \exists T_1, \ldots, T_n \in T \text{ such that } A_i = \bigcap_{j=1}^{n} T_j\}$$

Then $\mathcal{S}_T$ is contained in $\mathcal{O}_T$ because $\mathcal{O}_T$ is a topology. So it suffices to show that $\mathcal{S}_T$ is a topology as well. □

1.9. **Example** A metric space (M, d) gives rise to a topological space $(M, \mathcal{O}_d)$. A subbasis for the topology $\mathcal{O}_d$ on M is given by the open balls of radius $\epsilon \leq 1$ around arbitrary points:

$$\mathcal{S} = \{B_\epsilon(m) = \{x \in M \mid d(x,m) < \epsilon\} \quad | \quad m \in M, \epsilon < 1\}.$$

In fact, $\mathcal{S}$ is a basis, not only a subbasis and whenever $x \in U$ with $U \in \mathcal{O}_d$, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$ (Exercise). $\mathcal{O}_d$ is called the *metric topology* on M.

In particular, for any $n \geq 0$, euclidean space $\mathbb{R}^n$ with its norm induced metric $d(x, y) = \|x-y\|$ is a metric space. Moreover, any subset of a metric space is again a metric space (simply restrict the metric), and consequently any subset of $\mathbb{R}^n$ is a metric, and hence topological space. There are many different kinds of such subspaces, we consider the following two extreme cases:

(1) $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$, the n-dimensional sphere.
(2) The Cantor set $C \subseteq [0, 1]$ or $\mathbb{Q} \subseteq \mathbb{R}$.

We will later argue in which sense these examples are of different nature, based on the notion of connectedness (the sphere is connected, whereas the Cantor set and $\mathbb{Q}$ are totally disconnected).

1.10. **Remark** A topological space $(X, \mathcal{O})$ is called *metrizable*, if there is a metric d on X such that $\mathcal{O}$ is the metric topology of d. Not all topological spaces are metrizable as we discuss in an Exercise. Metrizable spaces will not play an important role in this course.

We come to the notion of continuous maps.

1.11. **Definition** Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. A map $f: X \to Y$ is called *continuous with respect to* $\mathcal{O}_X$ *and* $\mathcal{O}_Y$ if $f^{-1}(\mathcal{O}_Y) \subseteq \mathcal{O}_X$, that is, if for all $U \in \mathcal{O}_Y$, the preimage $f^{-1}(U)$ lies in $\mathcal{O}_X$. We denote the set of continuous maps from X to Y by $C((X, \mathcal{O}_X), (Y, \mathcal{O}_Y))$.

1.12. **Remark** Equivalently, f is continuous if $f^{-1}(A)$ is closed in X whenever A is closed in Y.

In what follows we will often write "Let X be a topological space". When doing so, we mean that X denotes a set equipped with a topology $\mathcal{O}$ which we leave implicit in the notation. In this case, we shall also simply say that $f: X \to Y$ is continuous and denote the set of continuous maps by $C(X, Y)$. We record the following obvious fact:

1.13. **Lemma** *Let $f: X \to Y$ and $g: Y \to Z$ be continuous maps between topological spaces. Then $g \circ f: X \to Z$ is continuous.*

Since also the identity of any topological space is continuous, we obtain a category Top whose objects are topological spaces and whose morphisms are continuous maps.

1.14. **Lemma** *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces and let $\mathcal{S}$ be a subbasis for $\mathcal{O}_Y$. Then f is continuous if and only if for any $U \in \mathcal{S}$, we have $f^{-1}(U) \in \mathcal{O}_X$.*

Proof. If f is continuous, then $f^{-1}(U) \in \mathcal{O}_X$ for all $U \in \mathcal{O}_Y$, so the only if follows from the inclusion $\mathcal{S} \subseteq \mathcal{O}_Y$. To see the converse, let $U \in \mathcal{O}_Y$. Since $\mathcal{S}$ is a subbasis, there is a set I and for each $i \in I$ there is a finite set J_i and a map $A: J_i \to \mathcal{S}$ such that

$$U = \bigcup_{i \in I} \bigcap_{j \in J_i} A_j.$$

Since taking inverse images commutes with unions and intersections, we obtain

$$f^{-1}(U) = \bigcup_{i\in I} \bigcap_{j\in J_i} f^{-1}(A_j).$$

By assumption, for all $i \in I$ and all $j \in J_i$, we have that $f^{-1}(A_j) \in \mathrm{O}_X$. Since O_X is a topology, we conclude that $f^{-1}(U) \in \mathrm{O}_X$ and hence f is continuous. □

Most likely you remember the ϵ-δ-criterion for continuity from Analysis. The following shows that the a priori different notions of continuous maps agree.

1.15. **Lemma** *Let (M, d) and (M', d') be metric spaces. A map $f\colon M \to M'$ is continuous with respect to the metric topologies if and only if f is ϵ-δ-continuous.*

Proof. We recall that f is called ϵ-δ-continuous, if for all $x \in M$ and all $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies that $d'(f(x), f(y)) < \epsilon$, in other words that $B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x)))$.

By Lemma 1.14, f is continuous if and only if $f^{-1}(B_\epsilon(z))$ is open for all $\epsilon > 0$ and $z \in M'$. This is only a condition when $f^{-1}(B_\epsilon(z))$ is non-empty. By the self-indexing trick of Observation 1.3 and the exercise alluded to in Example 1.9, f is therefore continuous if and only if for all $x \in f^{-1}(B_\epsilon(z))$, there exists $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(B_\epsilon(z))$, in other words, precisely when f is ϵ-δ-continuous. □

We come to first examples of constructing new topological spaces from old ones:

1.16. **Definition**
(1) Let (X, O) be a topological space and $i\colon Y \to X$ an injection. The restricted topology $\mathrm{O}_{|Y}$ on Y is given by the collection of sets $i^{-1}(U)$ for $U \in \mathrm{O}$.
(2) Let (X, O) be a topological space and $p\colon X \to Z$ a surjection. The quotient topology on Z is given by those sets $V \subseteq Z$ such that $p^{-1}(U)$ is open.

When i as above is the inclusion of a subset $Y \subseteq X$, we refer to the resulting topological space as as a *subspace* of (X, O).

1.17. **Lemma** *Let (X, O) be a topological space, $i\colon Y \to X$ an injection and $p\colon X \to Z$ a surjection. Then the restricted topology on Y and the quotient topology on Z are in fact topologies. Moreover, when (X', O') is another topological space, then $f\colon X' \to Y$ is continuous if and only if the composite $if\colon X' \to X$ is continuous. Likewise, $g\colon Z \to X'$ is continuous if and only if the composite $gp\colon X \to X'$ is continuous.*

Proof. That the putative topologies are indeed topologies follows from the fact that preimages commute with intersections and unions. If f and g are continuous, then so are if and gp. Let us conversely assume that if and gp are continuous. To see that f is continuous, let $V = i^{-1}(U)$ be an open of Y. Then $f^{-1}(V) = f^{-1}(i^{-1}(U)) = (if)^{-1}(U)$ which is open since U is and if is continuous. Likewise, let $U' \subseteq X'$ be open. Then $g^{-1}(U')$ is open if and only if $p^{-1}(g^{-1}(U'))$ is open which is the case because it is given by $(gp)^{-1}(U')$ and gp is continuous. □

1.18. **Corollary** *Let $f\colon X \to Y$ be a continuous map between topological spaces. Equip the image $f(X) \subseteq Y$ of f with the subspace topology. Then f restricts to a continuous surjective map $f\colon X \to f(X)$.*

1.19. **Remark** Let $p\colon X \to Z$ be a surjection. The relation $x \sim x'$ if $p(x) = p(x')$ is an equivalence relation on X, and the resulting map $X/\sim \to Z$ is a bijection. In particular, any surjection is given as the quotient of X by an equivalence relation. The above then shows that if X is a topological space equipped with an equivalence relation $\sim$ and given a continuous map $f\colon X \to Y$ which is compatible with the equivalence relation in the sense that $x \sim x'$ implies that $f(x) = f(x')$, the resulting map of sets $X/\sim \to Y$ is continuous with respect to the quotient topology on $X/\sim$. There are two typical examples for such equivalence relations to keep in mind:

(1) For $A \subseteq X$ a subset, the relation $x \sim x'$ if and only if x and x' are contained in A generates an equivalence relation. The quotient by this relation will be denoted X/A.
(2) For X a topological space, whose underlying set X is equipped with an action of a group G, the relation $x \sim x'$ if there exists $g \in G$ such that $gx = x'$ is an equivalence relation. The quotient will be denoted by X/G. In an exercise, we show that the quotient map $X \to X/G$ is open if the action of G on X is by continuous maps, i.e. that for all $g \in G$, the map $x \mapsto gx$ is continuous.

1.20. **Example** Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Consider the set

$$L(\mathbb{K}^{n+1}) = \{V \subseteq \mathbb{K}^{n+1} \mid V \text{ is a one-dimensional subspace of } \mathbb{K}^{n+1}\}$$

of lines in $\mathbb{K}^{n+1}$. Then $L(\mathbb{K}^{n+1})$ is the quotient of $\mathbb{K}^{n+1} \setminus \{0\}$ by the equivalence relation $x \sim x'$ if there exists $\lambda \in \mathbb{K}$ such that $x' = \lambda x$. Since $\mathbb{K} \cong \mathbb{R}^k$ for $k = 1, 2, 4$, we find that $\mathbb{K}^n \setminus \{0\}$ is canonically a topological space. With the quotient topology, $L(\mathbb{K}^{n+1})$ therefore is also a topological space, called *n-dimensional projective space* over $\mathbb{K}$, and is written $\mathbb{KP}^n$.

1.21. **Example** Let $n \geq 1$. Then the map $x \mapsto -x$ on $\mathbb{R}^{n+1}$ restricts to a self-map of S^n. This determines an action of C_2 on S^n. The quotient S^n/C_2 is homeomorphic to $\mathbb{RP}^n$, see **??**. A convenient way to prove this will be outlined later.

1.22. **Example** Actions on $\mathbb{R}^2$ by linear transformations: $\mathbb{Z}^2$ acts with quotient given by T^2. $\mathbb{Z} \rtimes \mathbb{Z}$ acts with quotient a Klein bottle. $\mathbb{Z}^2 \subseteq \mathbb{Z} \rtimes \mathbb{Z}$ is an index 2 subgroup. In particular, we have an induced map $\mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z} \rtimes \mathbb{Z}$.

1.23. **Example** Representations: Let $\rho\colon G \to U(n)$ be a unitary complex representation of a finite group. Then G restricts to an action on $S(\mathbb{C}^n)$. For instance one can look at $G = \mathbb{Z}/p$ a finite cyclic group of order p. Given q coprime to p, one can consider the automorphism $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ given by multiplication with q. One writes $\mathbb{C}(q)$ for the 1-dimensional representation given by the composite $\mathbb{Z}/p\mathbb{Z} \xrightarrow{\cdot q} \mathbb{Z}/p\mathbb{Z} \subseteq U(1)$. For $q_1, \ldots, q_n$ we obtain an action of $\mathbb{Z}/p\mathbb{Z}$ on $S(\mathbb{C}(q_1) \oplus \ldots \mathbb{C}(q_n))$. Exercise: This action is free (i.e. if $gx = x$ then $g = e$) if all q_i are coprime to p. The quotient space $L(p; q_1, \ldots, q_n)$ is called an $2n-1$-dimensional lens space. Lens spaces are examples of smooth manifolds (which we will not define in this course) which have a beautiful classification and history.

The question which finite groups can act freely on the sphere is a very intriguing one, and has been solved by a combination of a number of magical results. The whole problem is called the spherical space form problem. We will see the first tiny part in the course of these lectures: The only non-trivial finite group that can act freely on S^{2n} is the group C_2 (this uses an invariant called the Euler characteristic).

1.24. **Definition** Let X be a topological space. A set $\{U_i\}_{i\in I}$ of open sets of X is called an *open cover* of X, if $X = \bigcup_{i\in I} U_i$.

The following lemma says that continuity of a map $f\colon X \to Y$ is a local property on X:

1.25. **Lemma** *Let $f\colon X \to Y$ be a map between topological spaces and $\{U_i\}_{i\in I}$ an open cover of X. Then f is continuous if and only if the restrictions $f_{|U_i}\colon U_i \to Y$ are continuous for all $i \in I$, where we endow U_i with the subspace topology.*

Proof. If f is continuous, then so are all restrictions since the inclusion of an open set $U \subseteq X$ is continuous when U is endowed with the subspace topology. The converse follows from the observation that a set $A \subseteq X$ is open if (and only if) $A \cap U_i$ is open for all $i \in I$, which follows from the fact that $\bigcup_{i\in I} A \cap U_i = A \cap X = A$ together with the fact that unions of open sets are open. □

Lemma 1.25 also implies that for a topological space X, the association $U \mapsto C(U, Y)$ forms a sheaf of sets on X. If you have not heard about sheaves, just ignore that this is so.

1.26. **Definition** Let X and Y be topological spaces. A map $f\colon X \to Y$ is called a *homeomorphism*, if it is continuous and there exists a continuous map $g\colon Y \to X$ such that $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$.

By definition, a homeomorphism is precisely an isomorphism in the category Top.

1.27. **Remark** Equivalently, a homeomorphism is a continuous bijection $f\colon X \to Y$, whose set-theoretic inverse $f^{-1}\colon Y \to X$ is continuous. The latter is in turn equivalent to the condition that $f(U)$ is open whenever U is open. Such maps are called *open*. In other words, a homeomorphism is an open and continuous bijection.

1.28. **Example** A continuous bijection need not be a homeomorphism, contrary to what we are used to from linear algebra, where for instance a linear map of vector spaces which is bijective is an isomorphism. Indeed, consider a set X and a strict inclusion of topologies $\mathrm{O} \subseteq \mathrm{O}'$ on X, e.g. the indiscrete and discrete topology for a set with more than one point. Then the identity $(X, \mathrm{O}') \to (X, \mathrm{O})$ is continuous and bijective but not open.

1.29. **Example** Let $a < b$ be real numbers. Then the open interval $(a, b) \subseteq \mathbb{R}$ is homeomorphic to $\mathbb{R}$. Indeed, the map

$$f(x) = \tan\big(\frac{\pi(x-a)}{b-a} - \frac{\pi}{2}\big)$$

is a homeomorphism. Indeed, this map is the composite of an affine linear homeomorphism $(a, b) \to (-\frac{\pi}{2}, \frac{\pi}{2})$ and the homeomorphism $\tan\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$. Here, the inverse of tan is the function arctan, and the continuitiy of tan and arctan is usually shown in a first course in Analysis. We refrain from reproducing it here.

1.30. **Example** Similarly, let $n \geq 2$, $x \in \mathbb{R}^n$ and $\epsilon > 0$. Then $B_\epsilon(x)$ is homeomorphic to $\mathbb{R}^n$. To see this, by an affine linear homeomorphism we may assume that $x = 0$. Then we perform the function of the previous example radially.

1.2. **Basic constructions.** The purpose of this subsection is to show that the category Top is bicomplete, i.e. it admits all small colimits and limits. We will first show this abstract statement and then discuss several very important examples of such limits and colimits. See Appendix A for the basics in category theory we will use here. First, we record the following:

1.31. **Lemma** *The forgetful functor* Top $\to$ Set*, sending* (X, O) *to* X*, admits both a left and a right adjoint. In particular, it commutes with all limits and colimits.*

Proof. The associations $X \mapsto (X, \{\emptyset, X\})$ and $X \mapsto (X, \mathcal{P}(X))$ are readily seen to be left and right adjoints, respectively. □

Now let I be a small category and $X\colon I \to \mathrm{Top}$, $i \mapsto X_i$ a functor. Lemma 1.31 implies that the underlying set of a colimit or a limit of this functor is the colimit and limit of the underlying sets. In particular, to construct a colimit and limit of X, we really need to endow the colimit and limit of the underlying set with an appropriate topology. We begin with the most basic case, that of products and coproducts of topological spaces.

1.32. **Lemma** *Let* I *be a set and let* $X_i \in \mathrm{Top}$ *for all* $i \in I$*.*

(1) *The set of subsets of* $\coprod_{i\in I} X_i$ *given by*
$$\{\iota_j(U) \subseteq \coprod_{i\in I} X_i \mid U \in \mathrm{O}(X_j) \text{ for some } i \in I\}$$
for a subbasis of a topology which we call the coproduct topology*. The family of inclusions* $\iota_j\colon X_j \subseteq \coprod_{i\in I} X_i$ *is continuous and exhibits* $\coprod_{i\in I} X_i$*, equipped with the coproduct topology, as a coproduct of the* X_i*'s.*

(2) *The set of subsets of* $\prod_{i\in I} X_i$ *given by*
$$\{p_j^{-1}(U) \subseteq \prod_{i\in I} X_i \mid U \in \mathrm{O}(X_j) \text{ for some } j \in I\}$$
forms a subbasis for a topology which we call the product topology*. The family of projections* $p_j\colon \prod_{i\in I} X_i \to X_j$ *is continuous and exhibits* $\prod_{i\in I} X_i$*, equipped with the product topology, as a product of the* X_i*'s.*

Proof. (1). First we note that the open sets for the coproduct topology is given by the sets $V \subseteq \coprod_{i\in I} X_i$ such that $V \cap X_i \in \mathrm{O}(X_i)$ for all $i \in I$. Indeed, this is a topology, and every such V is the union $\cup_{i\in I} V \cap X_i$ and $V \cap X_i$ lies in the claimed subbasis. Moreover, the inclusions $\iota_j\colon X_j \to \coprod_{i\in I} X_i$ are continuous for all $j \in I$ by construction. We need to show that a map $f\colon \coprod_{i\in I} X_i \to Y$, corresponding to maps $f_j\colon X_j \to Y$ is continuous if and only if all f_i are continuous. First suppose that f is continuous. Then $f_j = \iota_j f$ is also continuous. Conversely, assume that all $f_j\colon X_j \to Y$ are continuous. For $U \in \mathrm{O}(Y)$, we have that $f^{-1}(V) \cap X_j = f_j^{-1}(U) \in \mathrm{O}(X_j)$. Hence, $f^{-1}(U)$ is open as needed.

(2). First, we note that the projections $p_j\colon \prod_{i\in I} X_i \to X_j$ are continuous by definition (open sets of X_j are mapped to elements of the subbasis). Again, we need to show that a map $f\colon Y \to \prod_{i\in I} X_i$, corresponding to a family of maps $f_j\colon Y \to X_j$ continuous if and only if all f_j are continuous. If f is continuous, then so is $f_j = p_j f$. Conversely suppose all f_j are continuous. By Lemma 1.14 it suffices to show that $f^{-1}(V)$ is open for $V = p_j^{-1}(U)$ for all $j \in I$ and all $U \in \mathrm{O}(X_j)$. But then $f^{-1}(V) = (p_j f)^{-1}(U)$ which is indeed open. □

1.33. **Remark** Unlike the case of the coproduct topology, where the defining subbasis is a basis and the coproduct topology can be described very explicitly, the defining subbasis for the product topology is not a basis. However, a concrete basis is given by "finite-block-open" subsets: That is, in finitely many coordinates, we choose an open subset, and in all others we take the whole space. As a result, both the inclusions $\iota_j \colon X_j \to \amalg_{i\in I} X_i$ and the projections $p_j \colon \prod_{i\in I} X_i \to X_j$ are open maps.

Exercise. Let I be a set. Show that there is a canonical homeomorphism $X \times I^\delta \cong \coprod_{i\in I} X$.

As a consequence of what we have done so far, we arrive at the following result. We will discuss examples right after the proof.

1.34. **Proposition** *The category* Top *is bicomplete.*

Proof. Let I be a small category and $X \colon I \to \mathrm{Top}$. Then X admits a colimit and a limit. □

Proof. Recall that the limit of the functor $I \to \mathrm{Top} \to \mathrm{Set}$ is given by

$$\lim_{i\in I} X_i = \{(x_i) \in \prod_{i\in I} X_i \mid \forall f \in \mathrm{Hom}_I(i,j) \text{ we have } X_f(x_i) = x_j\}.$$

We give this set the subspace topology Definition 1.16(1) of the product topology obtained in Lemma 1.32. The universal property of a limit then follows immediately from Lemma 1.17. Likewise, we have that the colimit of the functor $I \to \mathrm{Top} \to \mathrm{Set}$ is given by

$$\operatorname*{colim}_{i\in I} X_i = \Big(\coprod_{i\in I} X_i\Big)/(\forall f \in \mathrm{Hom}_I(i,j) \text{ we set } X_f(x_i) \sim x_j)$$

In particular, the colimit admits a canonical surjection from the coproduct. We then equip the colimit with the quotient topology Definition 1.16(2) of the coproduct topology obtained in Lemma 1.32. The universal property of a colimit is again immediate from Lemma 1.17. □

1.35. **Example** An important "indexing" category I is given by $\bullet \leftarrow \bullet \rightarrow \bullet$. It is made so such that a functor $X \colon I \to \mathrm{Top}$ is simply given by two maps with common domain: $Y \xleftarrow{f} X \xrightarrow{f'} Y'$. A colimit of such a diagram is called a pushout, we will sometimes denote it by $C(f, f')$. Concretely, we have

$$C(f, f') = (Y \coprod Y')/(f(x) \sim f'(x) \forall x \in X)$$

This is most geometric in case f and f' is the inclusion of a subspace in Y and Y'. In this case, we image that $C(f, f')$ is given by taking the disjoint union of Y and Y' and then glue them together along the common subset X. In general, a pushout is perhaps less geometric, but its universal property tells us precisely what it means to give a continuous map out of it: Any commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow \\ Y' & \longrightarrow & Z \end{array}$$

gives rise to a unique map from $C(f, f') \to Z$ making all induced diagrams commute.

As an exercise, calculate the limit of the diagram $Y \xleftarrow{f} X \xrightarrow{f'} Y'$, either by definition or by abstract means.

1.36. **Example** Let X be a topological space and $A \subseteq X$ a subset. Then the evident diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ * & \longrightarrow & X/A \end{array}$$

is a pushout diagram. Here, X/A is the quotient space from Remark 1.19. Exercise: what is $X/\varnothing$?

1.37. **Example** Let X be a topological space. Then the cone $\mathrm{C}(X)$ on X is given by the pushout

$$\begin{array}{ccc} X \times \{1\} & \longrightarrow & X \times [0,1] \\ \big\downarrow & & \big\downarrow \\ * & \longrightarrow & \mathrm{C}(X) \end{array}$$

Likewise, the suspension $\Sigma(X)$ of X is given by the pushout

$$\begin{array}{ccc} X \times \{0\} & \longrightarrow & \mathrm{C}(X) \\ \big\downarrow & & \big\downarrow \\ \mathrm{C}(X) & \longrightarrow & \Sigma(X). \end{array}$$

Exercise: It is also given by the pushout $* \amalg_X \mathrm{C}(X)$.

The following example will become somewhat relavant later. Exercise:

1.38. **Example** Let X be a topological space and for $i = 1, 2$, let $U_i \subseteq X_i \subseteq X$ with U_i open such that $U_1 \cup U_2 = X$. Then the diagram

$$\begin{array}{ccc} X_1 \cap X_2 & \longrightarrow & X_1 \\ \big\downarrow & & \big\downarrow \\ X_2 & \longrightarrow & X \end{array}$$

is a pushout.

1.39. **Example** For pointed spaces (X, x) and (Y, y) define their wedge product $(X, x) \vee (Y, y)$ as the pushout $X \amalg_* Y$. Note that the category of pointed spaces is the slice category $\mathrm{Top}_{*/}$. Hence, the wedge sum is the coproduct of pointed topological spaces.

1.40. **Example** For pointed spaces (X, x) and (Y, y), define their smash product $(X, x) \wedge (Y, y)$ to be the quotient $X \times Y / X \vee Y$. Here, the map along which we take the quotient is the universal map obtained via the commutative diagram

$$\begin{array}{ccc} * & \xrightarrow{(x,y)} & X \times \{y\} \\ \big\downarrow{\scriptstyle (x,y)} & & \big\downarrow \\ \{x\} \times Y & \longrightarrow & X \times Y \end{array}$$

The smash product endows the category Top_* of pointed topological spaces with a symmetric monoidal structure.

1.41. **Observation** There is a functor $\mathrm{Top} \to \mathrm{Top}_*$ given by $X_+ = X \mapsto X \amalg \{*\}$. It is a left adjoint to the forgetful functor. Exercise: It satisfies that there is a canonical homeomorphism $(X \times Y)_+ \cong X_+ \wedge Y_+$. In more categorical terms, the functor $\mathrm{Top} \to \mathrm{Top}_*$ is symmetric monoidal with respect to the cartesian product on Top and the smash product on Top_*.

1.42. **Example** The indexing category $I^{\mathrm{op}} = \bullet \to \bullet \leftarrow \bullet$ is equally important. A functor from it to Top is then given by two maps with common codomain: $X \xrightarrow{f} Y \xleftarrow{f'} X'$. A limit of this diagram is called a pullback, sometimes denoted by $L(f, f')$. Conceretely, we have

$$L(f, f') = \{(x, x') \in X \times X' \mid f(x) = f'(x')\}$$

The universal property then gives that any commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow f \\ X' & \xrightarrow{f'} & Y \end{array}$$

gives rise to a unique map $Z \to L(f, f')$ making all induced diagrams commute. As an exercise, calculate the colimit of the diagram $X \xrightarrow{f} Y \xleftarrow{f'} X'$.

Exercise. Let $A \subseteq B \subseteq X$. Then the canonical map $B/A \to X/A$ is injective and the topology on B/A is the subspace topology. Moreover, $(X/A)/(B/A)$ is canonically homeomorphic to X/B.

1.3. Connected spaces.

1.43. **Definition** A non-empty topological space X is called

(1) *connected*, if the only non-empty open and closed subset of X is X itself.
(2) *locally connected*, if for all $x \in X$ and all open sets U containing x, there is a connected and open subspace $x \in V \subseteq U$.
(3) *weakly locally connected*, if for all $x \in X$, there exists a connected and open subspace $x \in V$.
(4) *path-connected*, if for any $x, y \in X$, there exists a continuous map $f: [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.
(5) *locally path-connected*, if for all $x \in X$ and all open sets U containing x, there is a path-connected open subspace $x \in V \subseteq U$.
(6) *weakly locally path-connected*, if for all $x \in X$, there exists a path-connected and open subspace $x \in V$.

1.44. **Remark** (1) A (path)-connected space X is in particular weakly locally (path)-connected: simply choose X as (path)-connected open subspace V. A space is locally (path)-connected spaces if and only if any open subset is weakly locally (path)-connected.
(2) A locally (path)-connected space need not be (path)-connected: Consider the coproduct of two (path)-connected spaces.

To shorten notation we will refer to spaces which are both locally (path)-connected *and* (path)-connected as *locally and globally* (path)-connected spaces.

1.45. **Example** For all $n \geq 0$ euclidean space $\mathbb{R}^n$ is locally and globally path-connected. As a result, any space which is locally euclidean, i.e. which admits an open cover U_i with U_i homeomorphic to $\mathbb{R}^n$ for some n, is locally path-connected. A topological manifold is a topological space which is locally euclidean in this sense plus some further assumptions which we have not defined yet. But it follows that any topological manifold including any smooth manifold (whatever precisely this is) is a locally path-connected space. Another important class of locally path-connected spaces are CW-complexes, which we will introduce later (these are not locally euclidean in the above sense in general).

1.46. **Definition** A connected component of X at a point $x \in X$ is a maximal connected subspace containing x, there is a unique such which we denote by $C(x)$, see Lemma 1.55 below. A connected component of X is simply a maximal connected subspace. We write $\pi(X)$ for the set of connected components of X.

Clearly, X is the set theoretic union of all connected components of X.

1.47. **Remark** A topological space is connected if it has exactly one connected component. Indeed, the map $x \mapsto C(x)$ is a surjection from $X \to \pi(X)$. We conclude that $\pi(\varnothing) = \varnothing$, so $\varnothing$ has 0 connected components and is therefore not connected. This is to be seen in analogy to the fact that 1 is not a prime number: Prime numbers are the natural numbers which are divisible by precisely 2 numbers.

1.48. **Definition** The relation on a topological space X given by $x \sim x'$ if and only if there exists a continuous map $f\colon [0,1] \to X$ (henceforth called a path) with $f(0) = x$ and $f(1) = x'$ is an equivalence relation on X. We denote by $\pi_0(X)$ the set of equivalence classes, and call the equivalence classes the path-connected components of X.

Exercise. Show that $X \mapsto \pi_0(X)$ is a functor Top $\to$ Set. Show that it commutes with products.

1.49. **Observation** Let $f\colon X \to Y$ be a homeomorphism. Then X is (locally) (path) connected if and only if Y is.

As a consequence of the above, we can deduce that many spaces are not homeomorphic:

1.50. **Proposition** *The spaces* $[0,1], [0,1[$*, and* $(0,1)$ *and* $\mathbb{R}^n$ *for* $n \geq 2$ *are pairwise non-homeomorphic.*

Proof. We begin to show that $[0,1]$ is not homeomorphic to any of the other spaces, here denoted X. Indeed, assume a given homeomorphism $f\colon [0,1] \to X$. Then $(0,1)$ is homeomorphic to $X \setminus \{x_0, x_1\}$ for $x_0 \neq x_1$ elements of X. Since $(0,1)$ is path-connected, so is $X \setminus \{x_0, x_1\}$. However, removing two distinct points from $[0,1[$ or $(0,1)$ results in a non path-connected space. This shows that $[0,1]$ is not homeomorphic to $[0,1[$ or $(0,1)$. Likewise $[0,1] \setminus \{\frac{1}{2}\}$ is not path-connected. However, for $n \geq 2$, $\mathbb{R}^n \setminus \{x\}$ is path connected for any $x \in \mathbb{R}^2$: Pick any two points $y, z \in \mathbb{R}^n \setminus \{x\}$. If the straight line is contained in $\mathbb{R}^n \setminus \{x\}$ we are done. If not, pick any point w outside the straight line. Then the straigth line between

w and y as well as between w and z do not contain x. Again, we are done. Likewise $[0,1]$ is not homeomorphic to $(0,1)$ since $[0,1[\setminus\{0\}$ is path-connected, unlike $(0,1)\setminus\{x\}$, and it is not homeomorphic to $\mathbb{R}^n$ because $[0,1[\setminus\{\frac{1}{2}\}$ is not path-connected. The same argument shows that $(0,1)$ is not homeomorphic to $\mathbb{R}^n$. □

We begin with the following important basic example of a connected space:

1.51. **Lemma** *The interval* $[0,1]$ *is connected.*

Proof. Let $A\subseteq[0,1]$ be non-empty open and closed and assume without loss of generality that $0\in A$ (else replace A by $[0,1]\setminus A$). Let $\alpha=\sup\{x\in A\}$. Since A is closed, we know from Analysis that $\alpha\in A$. Assume that $\alpha<1$. Then since A is open and ϵ-balls form a basis of open sets of $[0,1]$, we deduce that there exists an $\epsilon>0$ such that $B_\epsilon(\alpha)\subseteq A$. This implies that $\alpha+\frac{\epsilon}{2}\in A$, contradicting the defining property of α. Hence $\alpha=1$ and $A=[0,1]$ as needed. □

We can use this result to produce many examples of connected spaces:

1.52. **Corollary** *A path-connected space is connected. In particular*

(1) *A weakly locally path-connected space is weakly locally connected, and*
(2) *A locally path-connected space is locally connected.*

Proof. We show that a non-connected space X is not path-connected. This is clear if $X=\varnothing$. Else, write $X=A\cup X\setminus A$ with A closed and open and different from $\emptyset$ and X. Given a continuous map $f\colon[0,1]\to X$ with $f(0)\in A$, we find that $f^{-1}(A)$ is a non-empty closed and open subset of $[0,1]$ and hence equal to $[0,1]$. We deduce from Lemma 1.51 that $f(1)\in A$, showing that there is no continuous path from elements in A to elements in $X\setminus A$. The "in particular" follows readily. □

1.53. **Remark** The argument above shows that two points x,x' which lie in the same path component must also lie in the same component. Therefore, there is a canonical surjection $\pi_0(X)\to\pi(X)$ from the path-connected components of X to the connected components.

1.54. **Lemma** *Let* $f\colon X\to Y$ *be a continuous map and* $A\subseteq X$ *a connected subspace. Then* $f(A)\subseteq Y$ *is also connected (in the subspace topology).*

Proof. Let $B\subseteq f(A)$ be a non-empty, open and closed subspace of $f(A)$. Recall that $f\colon A\to f(A)$ is continuous (as follows from the definition of the subspace topology). Therefore $f^{-1}(B)$ is open and closed in A, and it is non-empty since B is non-empty. Hence, $f^{-1}(B)=A$ since A is connected. But then $B=f(f^{-1}(B))=f(A)$ and therefore, $f(A)$ is connected. □

Next, we investigate connected components of a topological space.

1.55. **Lemma** *The union of all connected subspaces containing* x *is connected. It is therefore the (unique) connected component containing* x*, we write* $C(x)$ *for it.*

Proof. Let A be a non-empty open and closed subset of the union T of all connected subspaces C containing x. As A is non-empty, there is such C with $C\cap A\neq\varnothing$ and this intersection is a closed and open subset of C. Since C is connected, we find $C\cap A=C$ and therefore that $C\subseteq A$. In particular, we have that $x\in A$. We conclude that the intersection $C'\cap A$ of *every* connected subspace C' containing x is non-empty (as it contains x) and therefore, by the same reasoning as before, $C'\subseteq A$. We conclude that $T\subseteq A$. Moreover, by definition $A\subseteq T$ and therefore $A=T$. We conclude that T is connected. □

1.56. **Lemma** *Let X be a topological space and $x, y \in X$. If $y \in C(x)$, then $C(y) = C(x)$. In particular, the relation $x \simeq y$ if and only if $y \in C(x)$ is an equivalence relation, and the set of equivalence classes is $\pi(X)$.*

Proof. Since $y \in C(x)$ and $C(x)$ is connected, we find that $C(x) \subseteq C(y)$ by Lemma 1.55 and hence $x \in C(y)$. The same argument with roles of x and y reversed shows that also $C(y) \subseteq C(x)$. The "in particular" follows readily. □

1.57. **Definition** Let $Y \subseteq X$ be a subspace. The closure of Y, denoted $\overline{Y}$, is given by

$$\overline{Y} = \bigcap \{A \subseteq X \mid A \text{ closed and } Y \subseteq A\}.$$

Likewise we define the open interior of Y, denoted $\mathring{Y}$, by

$$\mathring{Y} = \bigcup \{U \subseteq X \mid U \text{ open and } U \subseteq Y\}.$$

The open interior will only play a role later in the course.

1.58. **Lemma** *Let $C \subseteq X$ be a connected subspace. Then $\overline{C}$ is also connected.*

Proof. First we note the following: Let $U \subseteq X$ be an open subspace and $C \subseteq X$ any subspace. If $U \cap C = \varnothing$, then $U \cap \overline{C} = \varnothing$ as well. Indeed, the assumption implies that $C \subseteq X \setminus U$ and $X \setminus U$ is closed. The definition of $\overline{C}$ then implies that $\overline{C} \subseteq X \setminus U$ as well. Therefore $U \cap \overline{C} = \varnothing$ as claimed.

Now let C be connected and $A \subseteq \overline{C}$ non-empty, open and closed. Then $A \cap C$ is open and closed in C and non-empty by what we just argued. We deduce that $A \cap C = C$ since C is connected, and therefore that $C \subseteq A$. By definition of $\overline{C}$ we then find that $\overline{C} \subseteq A$, since A is closed. Therefore $A = \overline{C}$ and hence $\overline{C}$ is connected. □

1.59. **Lemma** *Let X be a space.*

1. *Connected components of X are closed.*
2. *If X is weakly locally connected, then connected components of X are open.*

In particular, the inclusions of the connected components of a weakly locally connected space exhibit X as the coproduct over its connected components.

Proof. (1). Let C be a connected component of X. Then $C \subseteq \overline{C}$ and $\overline{C}$ is again connected by Lemma 1.58. Since C is a maximal connected subspace, $C = \overline{C}$ and is therefore closed. (2). Let C be a connected component. For any $x \in C$, we deduce that $C = C(x)$. Since X is weakly locally connected, there is an open and connected set V containing x. By Lemma 1.55, $V \subseteq C(x) = C$. By the self-indexing trick, C is open. The "in particular" follows from Exercise 5 Sheet 1. □

1.60. **Remark** If X is such that $\pi(X)$ is a finite set, then it follows directly that all components of X are also open (because the complement is a finite union of closed sets). We deduce that X is weakly locally connected, because one can choose the connected component of any point as the required V. In particular, part (2) of Lemma 1.59 is in fact an "if and only if".

1.61. **Corollary** *Let X be a weakly locally path connected space. Then the map $\pi_0(X) \to \pi(X)$ is a bijection, i.e. the path-connected components are precisely the connected components.*

Proof. X is weakly locally connected by Corollary 1.52. Hence by Lemma 1.59, X is the coproduct of its components. Moreover, as an exercise, we showed that the canonical map

$\coprod_{i\in I}\pi_0(X_i) \to \pi_0(\coprod_{i\in I} X_i)$ is a bijection. Hence, it suffices to show that a weakly locally path-connected space which is connected is in fact path-connected. To do so, let $x \in X$. Consider the set $A = \{y \in X \mid \exists f\colon [0,1] \to X \text{ such that } f(0) = x, f(1) = y\}$. Then A contains x. Moreover, since X is weakly locally path-connected this space is open and closed. Indeed, for $y \in X$, there exists an open and path-connected neighborhood V_y of y. Hence, if $y \in A$, we find $V_y \subseteq A$ showing that A is open by the self-indexing trick. Likewise, if $y \notin A$, we find $V_y \cap A = \varnothing$, showing that A is also closed. Since X is connected, $A = X$ and therefore X is path-connected. □

We briefly mention the drastic opposite of connected spaces (those where $C(x) = X$ for all $x \in X$), the totally disconnected spaces:

1.62. **Definition** A topological space X is called totally disconnected if for all $x \in X$, we have $C(x) = \{x\}$, or equivalently, if every connected subspace of X is of the form $\{x\}$ for some $x \in X$.

1.63. **Example** The Cantor set and $\mathbb{Q}$ are totally disconnected. Any discrete space is totally disconnected. The p-adic integers $\mathbb{Z}_p$ with their canonical inverse limit topology are totally disconnected. Indeed, it follows from Exercise ? Sheet 3 that totally disconnected spaces are closed under limits in topological spaces, that is, given a functor $I \to \mathrm{Top}$ taking values in totally disconnected spaces, the limit is again totally disconnected. One can also prove this directly by showing that: Products of totally disconnected spaces are totally disconnected and subspaces of totally disconnected spaces are totally disconnected. In particular, profinite spaces (inverse limits of discrete finite spaces) are totally disconnected. Profinite topologies arise in number theory in various ways: On the one hand side, Galois groups of non-finite Galois extensions are canonically profinite groups (and in this case, the Galois correspondence needs to take the profinite topology into account) and on the other hand, via the notion of completions of commutative rings at places or prime ideals. A special case of this is how one arrives at $\mathbb{Z}_p$ and similar topological rings like $\mathbb{Q}_p$.

1.4. Compact spaces.

1.64. **Definition** A topological space X is called compact, if for every open cover $X = \bigcup_{i\in I} U_i$ there exists a finite subcover, that is, a finite subset $J \subseteq I$ such that $X = \bigcup_{j\in J} U_j$. It is called

1. *locally compact* if for every $x \in X$ and open set $x \in U \subseteq X$, there is an open subset V and a compact subset K such that $x \in V \subseteq K \subseteq U$.
2. *weakly locally compact* if for every $x \in X$, there exists an open subset V and a compact subset K such that $x \in V \subseteq K$.

1.65. **Remark** As in the case of connected spaces, a compact space X is weakly locally compact. However, it need not be locally compact. We will discuss an example on an exercise sheet.

1.66. **Lemma** (1) *Let $f\colon X \to Y$ be a continuous map and $K \subseteq X$ a compact subspace. Then $f(K) \subseteq Y$ is compact.*
(2) *Closed subspaces of compact spaces are compact.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of $f(K)$. Since $f\colon K \to f(K)$ is continuous, $\{f^{-1}(U_i)\}_{i\in I}$ forms an open cover of K. Since K is compact, there exists a finite subset $J \subseteq I$ so that $K = \bigcup_{j\in J} f^{-1}(U_j)$. Then we find $f(K) \subseteq f(\bigcup_{j\in J} f^{-1}(U_j)) = \bigcup_{j\in J} U_j$ proving (1). To see (2), let $A \subseteq X$ be a closed subspace of a compact space X. Let $\{U_i\}_{i\in I}$ be an open cover of A. Then $\{U_i\}_{i\in I} \cup \{X \setminus A\}$ is an open cover of X. There is then a finite subcover given by a finite subset $J \in I$ and we find that $\{U_j\}_{j\in J}$ is a finite subcover, showing that A is compact. □

The following is an equivalent formulation of compactness which we will use later.

1.67. **Lemma** *Let X be a compact space, $\{Z_i\}_{i\in I}$ a collection of closed subspaces such that each intersection of finitely many Z_i's is non-empty. Then the intersection of all Z_i's is non-empty.*

Proof. Exercise. □

An important theorem about compact spaces is Tychonoff's theorem.

1.68. **Theorem** *An arbitrary product of compact spaces X_i is compact. If all X_i are non-empty, the converse holds as well.*

To prove this, we will make use of the notion of (ultra)filters.

1.69. **Definition** A filter $\mathcal{F}$ on a set X is a collection of subsets of X, i.e. $\mathcal{F} \subseteq \mathcal{P}(X)$, satisfying the following axioms:

(1) $X \in \mathcal{F}$ and $\varnothing \notin \mathcal{F}$,
(2) if $U \in \mathcal{F}$ and $U \subseteq V$, then $V \in \mathcal{F}$.
(3) if $U_1, \dots, U_n \in \mathcal{F}$, then $U = \bigcap U_i \in \mathcal{F}$.

We write $\mathcal{F} \subseteq \mathcal{F}'$ if $U \in \mathcal{F}$ implies that $U \in \mathcal{F}'$. An ultrafilter is a maximal filter with respect to $\subseteq$.

1.70. **Lemma** *Let $\mathcal{A}$ be a collection of subsets of a set X such that each finite subcollection has non-empty intersection. Then*

$$\langle \mathcal{A} \rangle = \{B \subseteq X \quad | \quad \exists A_1, \dots, A_n \in \mathcal{A}, A_1 \cap \dots \cap A_n \subseteq B\}$$

is a filter (the smallest filter containing $\mathcal{A}$). We call $\langle \mathcal{A} \rangle$ the filter generated by $\mathcal{A}$.

Proof. The axioms of a filter are immediate. It is also clear that $\langle \mathcal{A} \rangle$ is the smallest filter containing $\mathcal{A}$. □

1.71. **Remark** If $\mathcal{A}$ is a collection of subsets such that there is a finite subcollection with empty intersection, then $\mathcal{A}$ is not contained in any filter (else that filter would contain the empty-set). The assumption in Lemma 1.70 can therefore not be dropped.

1.72. **Lemma** *Let X be a set.*

(1) *A filter $\mathcal{F}$ on X is an ultrafilter if and only if for all $A \subseteq X$, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.*
(2) *For an ultrafilter $\mathcal{F}$ we have the following. If $A \cup B \in \mathcal{F}$, then $A \in F$ or $B \in \mathcal{F}$.*

Proof. (1) Let $\mathcal{F}$ be an ultrafilter and let $A \subseteq X$ so that $X \setminus A \notin \mathcal{F}$. We aim to show that $A \in \mathcal{F}$. Now, for $B \in \mathcal{F}$ we find that B is not a subset of $X \setminus A$ (else $X \setminus A \in \mathcal{F}$). Hence, $B \cap A \neq \varnothing$ for all $B \in \mathcal{F}$. Consequently, the collection of subset $\mathcal{F} \cup \{A\}$ has the property that each finite subcollection has non-empty intersection. By Lemma 1.70, it generates a filter $\mathcal{F}'$ with $\mathcal{F} \subseteq \mathcal{F}'$. Since $\mathcal{F}$ is an ultrafilter, we find $\mathcal{F} = \mathcal{F}'$ and therefore that $A \in \mathcal{F}$ (since $A \in \mathcal{F}'$).

Conversely, let us assume that $\mathcal{F}$ is a filter such that for all $A \subseteq X$, either A or $X \setminus A$ is contained in $\mathcal{F}$. Let $\mathcal{F} \subseteq \mathcal{G}$ for some filter $\mathcal{G}$ and $A \in \mathcal{G}$. We deduce that $X \setminus A \notin \mathcal{G}$ and therefore also that $X \setminus A \notin \mathcal{F}$. Hence $A \in \mathcal{F}$ and therefore $\mathcal{F} = \mathcal{G}$. Consequently, $\mathcal{F}$ is an ultrafilter.

(2) Let $A \cup B \in \mathcal{F}$ and $A \subseteq X$. If $A \in \mathcal{F}$ we are done. If $A \notin \mathcal{F}$, then $X \setminus A \in \mathcal{F}$ since $\mathcal{F}$ is an ultrafilter. We also have

$$B \setminus A = X \setminus A \cap (A \cup B).$$

Therefore, $B \setminus A \in \mathcal{F}$. Since $B \setminus A \subseteq B$, we find that $B \in \mathcal{F}$. □

1.73. **Lemma** *Every filter $\mathcal{F}$ is contained in an ultrafilter $\bar{\mathcal{F}}$.*

Proof. This follows from Zorn's Lemma, as soon as we show that the filtered colimit $\mathcal{F} = \operatorname{colim}_{i \in I} \mathcal{F}_i$ of filters $\mathcal{F}_i$, indexed over a totally ordered set I, with injective transition maps, i.e. where for $i \leq i'$ the map $\mathcal{F}_i \to \mathcal{F}'_i$ is an inclusion of filters, is again a filter. This, however, is again immediate from the definitions. □

1.74. **Definition** Let $f\colon X \to Y$ a map of sets and $\mathcal{G}$ a collection of subsets of X. We denote by $f_*(\mathcal{G})$ the collection of subsets $B \subseteq Y$ such that $f^{-1}(B) \in \mathcal{G}$.

1.75. **Lemma** *Let $f\colon X \to Y$ be a map of sets and $\mathcal{F}$ an (ultra)filter on X. Then $f_*(\mathcal{F})$ is an (ultra)filter on Y.*

Proof. From $f^{-1}(Y) = X$ and $f^{-1}(\varnothing) = \varnothing$ we get axiom (1). Now let $B \in f_*(\mathcal{F})$ and $B \subseteq B'$. Then $f^{-1}(B) \subseteq f^{-1}(B')$ so that $f^{-1}(B') \in \mathcal{F}$ and therefore $B' \in f_*(\mathcal{F})$, giving axiom (2). Likewise, for $B_1, \dots, B_n \in f_*(\mathcal{F})$. Since $f^{-1}(\bigcap_i B_i) = \bigcap_i f^{-1}(B_i)$, we find that $\bigcap_i B_i \in f_*(\mathcal{F})$, giving axiom (3). Assume now that $\mathcal{F}$ is in addition an ultrafilter. By Lemma 1.72 it suffices to show that for $B \subseteq Y$, we have $B \in f_*(\mathcal{F})$ or $Y \setminus B \in f_*(\mathcal{F})$. By definition this means that $f^{-1}(B) \in \mathcal{F}$ or $f^{-1}(Y \setminus B) \in \mathcal{F}$ which is true since $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$ and $\mathcal{F}$ is an ultrafilter. □

1.76. **Definition** Let X be a topological space and $x \in X$ a point. The neighborhood filter $\mathcal{U}(x)$ consists of the neighborhoods of x, that is all subsets $N \subseteq X$ with $x \in N$ and such that there exists an open $U \subseteq N$ with $x \in U$ - this is also the filter generated by the collection $\mathcal{A}$ of open subsets containing x. We say that a filter $\mathcal{F}$ converges to a point x if $\mathcal{U}(x) \subseteq \mathcal{F}$. We then also say that x is a limit point of $\mathcal{F}$ and that $\mathcal{F}$ is convergent.

Compact spaces can be characterized using filters as follows.

1.77. **Theorem** *A topological space is compact if and only if every ultrafilter has at least one limit point.*

Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X and assume that every ultrafilter on X has at least one limit point. Consider the set $\mathcal{A} = \{A_i\}_{i \in I}$ with $A_i = X \setminus U_i$. If there is no finite subcover of $\mathcal{U}$, then the collection $\mathcal{A}$ has the property that every finite intersection of elements in $\mathcal{A}$ is non-empty by the argument used in the proof of Lemma 1.67. Consequently, $\mathcal{A}$ is contained in an ultrafilter $\mathcal{F}$, by Lemma 1.70 and Lemma 1.73. $\mathcal{F}$ then has a limit point x for which we can find $i \in I$ such that $x \in U_i$. Since $\mathcal{F}$ converges to x, we have $U_i \in \mathcal{F}$. However, by construction also $A_i \in \mathcal{F}$ which is a contradiction, so a finite subcover of $\mathcal{U}$ must exist, hence X is compact.

Conversely, let $\mathcal{F}$ be an ultrafilter on X which has no limit point. For $x \in X$ consider an open U_x which is not contained in $\mathcal{F}$. Then $\mathcal{U} = \{U_x\}_{x\in X}$ is an open cover of X. Suppose there is a finite subcover indexed by $x_1, \dots, x_n$, then we obtain $U_{x_1} \cup \cdots \cup U_{x_n} = X \in \mathcal{F}$. By Lemma 1.72 part (2) we deduce that there is an $i \in \{1, \dots, n\}$ such that $U_{x_i} \in \mathcal{F}$, again a contradiction. Therefore, there is no finite subcover of $\mathcal{U}$ and consequently, X is not compact. □

Exercise. Let X be a topological space and $A \subseteq X$ a subset. Show that a point x lies in $\overline{A}$ if and only if there is a Filter $\mathcal{F}$ on A which converges to x in the sense that $\mathcal{U}(x) \cap A \subseteq \mathcal{F}$.

We now come to the proof of Tychonoff's theorem.

Proof of Theorem 1.68. Let $\{X_i\}_{i\in I}$ be a collection of compact spaces. We wish to show that $\prod_I X_i$ is compact. So let $\mathcal{F}$ be an ultrafilter in $\prod_I X_i$ and let $p_j\colon \prod_I X_i \to X_j$ be the projection. By Lemma 1.75, $(p_j)_*(\mathcal{F})$ is an ultrafilter on X_j, so there is a limit point $x_j \in X_j$ for $(p_j)_*(\mathcal{F})$ by Theorem 1.77. We claim that the sequence $x = (x_j)_{j\in J}$ is a limit point of $\mathcal{F}$ (to form this sequence, we make use of the axiom of choice). For this, it suffices to show that any element U of a subbasis of the topology on $\prod_I X_i$ with $x \in U$ is contained in $\mathcal{F}$. A subbasis is given by $p_j^{-1}(U_j)$ for $j \in I$ and $U_j \subseteq X_j$ open. So we need to show that $p_j^{-1}(U_j) \in \mathcal{F}$ which is the case if and only if $U_j \in (p_j)_*(\mathcal{F})$ by definition of $(p_j)_*(\mathcal{F})$. This is the case by the assumption that x_j is a limit point of $(p_j)_*(\mathcal{F})$. □

1.78. **Addendum** Let X and Y be locally compact. Then also $X \times Y$ is locally compact. Indeed, pick $(x, y) \in X \times Y$ and $U \subseteq X \times Y$ which contains (x, y). Then by the definition of the product topology there are open sets $U_x \subseteq X$ containing x and $U_y \subseteq Y$ containing y such that $U_x \times U_y \subseteq U$. Since X and Y are locally compact, we can find $x \in V_x \subseteq K_x \subseteq U_x$ and $y \subseteq V_y \subseteq K_y \subseteq U_y$ such that V_x, V_y are open and K_x, K_y are compact. Then we have $(x, y) \in V_x \times V_y \subseteq K_x \times K_y \subseteq U_x \times U_y \subseteq U$ and $V_x \times V_y$ is open and $K_x \times K_y$ is compact by Tychonoff. Consequently, $X \times Y$ is locally compact.

1.5. **Hausdorff spaces.**

1.79. **Definition** A topological space is called

(1) Hausdorff (T2), if for all $x \neq x' \in X$ there are disjoint open sets U, U' with $x \in U$ and $x' \in U'$.
(2) regular (T3), if for all closed subsets $A \subseteq X$ and $x \in X \setminus A$, there are disjoint open sets U, V with $A \subseteq U$ and $x \in V$.
(3) normal (T4), if for all $A, A' \subseteq X$ disjoint closed, there are disjoint open sets U, U' with $A \subseteq U$ and $A' \subseteq U'$.

1.80. **Remark** Sometimes, it is required that normal and regular spaces are Hausdorff.

1.81. **Example** Any metrizable space is Hausdorff. Every subspace of a Hausdorff space is Hausdorff. In particular, every subspace of euclidean space $\mathbb{R}^n$ is Hausdorff. Quotients of Hausdorff spaces are not Hausdorff, for instance $[0, 1]/[0, 1)$ is not Hausdorff. see e.g. Exercise 4 Sheet 1.

1.82. **Example** Any bounded and closed subset of $\mathbb{R}^n$ is a compact Hausdorff space. Compactness indeed follows from the Heine-Borel property of $\mathbb{R}^n$. In particular, the interval $[0,1]$ and the cube $[0,1]^n$ are compact Hausdorff spaces, as is the Cantor set $C \subseteq [0,1]$.

Exercise. A space X is Hausdorff if and only if the diagonal $\Delta(X) \subseteq X \times X$ is closed. More generally, given a map $p\colon X \to Y$, show that the diagonal $\Delta(X) \subseteq X \times_Y X$ is closed if and only if for all $y \in Y$ and all $x, x' \in p^{-1}(y)$ with $x \neq x'$ there are disjoint open subsets U_x and U_x' of X containing x and x', respectively.

Hausdorff spaces, just as compact spaces, can be characterized using filters.

1.83. **Theorem** *A topological space X is Hausdorff if and only if every Filter $\mathcal{F}$ on X has at most one limit point.*

Proof. Suppose X is Hausdorff. Then for $x \neq y$, there are disjoint open sets U_x and U_y containing x and y, respectively. Hence, if x is a limit point of $\mathcal{F}$ we have $U_x \in \mathcal{F}$. Therefore, $U_y \notin \mathcal{F}$ (else $\varnothing \in \mathcal{F}$) and hence y is not a limit point of $\mathcal{F}$. Conversely, suppose X is not Hausdorff and pick x and y such that any two opens around x and y have non-trivial intersection. Then the collection $\{U \subseteq X \mid U \text{ open and } x \in U \text{ or } y \in U\}$ generates a filter by Lemma 1.70. This filter has both x and y as limit points. □

1.84. **Corollary** *A topological space is compact Hausdorff if and only if every Ultrafilter converges to precisely one limit point.*

Proof. The only if is immediate from Theorem 1.77 and Theorem 1.83. For the "if" part, it remains to show that any filter converges to at most one limit point. However, any limit point of a filter is also a limit point of any ultrafilter it is contained in. So there is at most one limit point for any filter. □

1.85. **Lemma** *Let X be a Hausdorff space.*

(1) *For $K \subseteq X$ compact and $y \in X \setminus K$, there are disjoint open sets $U \supseteq K$ and $V \ni y$.*
(2) *Compact subsets of X are closed.*

In particular, points are closed in X.

Proof. Let $K \subseteq X$ be compact. Since X is Hausdorff, for $y \in X \setminus K$ and $x \in K$ we can find disjoint open sets $U_x \ni x$ and $V_{y,x} \ni y$. Then $\{U_x\}_{x \in K}$ is an open cover of K. Since K is compact we can find a finite subcover, indexed by $x_1, \ldots, x_n$. Set $V = V_{y,x_1} \cap \cdots \cap V_{y,x_n}$ and $U = U_{x_1} \cup \cdots \cup U_{x_n}$. Then $U \cap V = \varnothing$, $K \subseteq U$ and $y \in V$, showing (1). For (2), let $K \subseteq X$ be compact. For $y \notin K$ we find $V_y \ni y$ and $U \subseteq K$ with U, V open and $U \cap V = \varnothing$, in particular $V_y \cap K = \emptyset$. But then we have $X \setminus K = \bigcup_{y \in X \setminus K} V_y$ so that K is closed. Clearly, points are compact and hence closed by (2). □

1.86. **Corollary** *A finite space is Hausdorff if and only if it is discrete.*

1.87. **Lemma** *Compact Hausdorff spaces are normal.*

Proof. Let $A, A' \subseteq X$ be disjoint closed subsets and let $x \in A$. Since A' is closed and X is compact, A' is also compact by Lemma 1.66. By part (1) of Lemma 1.85 there are disjoint opens $V_x \supseteq A'$ and $U_x \ni x$. We then get that $A \supseteq \bigcup_{x \in A} U_x$, and since A is also compact, there is again a finite subcover indexed by $x_1, \ldots, x_n \in A$, so we have $A \subseteq U_{x_1} \cup \cdots \cup U_{x_n} = U$. Defining similarly as before $V = V_{x_1} \cap \cdots \cap V_{x_n}$ we find that $A' \subseteq V$ and $U \cap V = \varnothing$. □

Later we will make use of the following result.

1.88. **Corollary** *Compact Hausdorff spaces are locally compact.*

Proof. Let X be a compact Hausdorff space and $U \subseteq X$ an open subset containing a point x. Since X is normal, the two closed subsets $\{x\}$ and $X \setminus U$ can be separated by disjoint open sets V_1 containing x and V_2 containing $X \setminus U$. Then $x \in V_1 \subseteq X \setminus V_2 \subseteq U$. Since $X \setminus V_2$ is closed and X is compact, $X \setminus V_2$ is itself compact, so X is indeed locally compact. □

1.89. **Lemma** *Let $f\colon X \to Y$ be a continuous map with X compact and Y Hausdorff.*

(1) *The map f is closed.*
(2) *If f is surjective, it is a quotient map.*
(3) *If f is bijective, it is a homeomorphism.*

In other words, (3) implies that the forgetful functor $\mathrm{CH} \to \mathrm{Set}$ *is conservative.*

Proof. (1) Let $A \subseteq X$ be closed. By Lemma 1.66 part (2), A is compact. Hence by Lemma 1.66 part (1), $f(A)$ is also compact, and hence by Lemma 1.85, $f(A)$ is closed. (2) We recall that f is a quotient map if a subset $U \in Y$ is open if and only if $f^{-1}(U) \subseteq X$ is open. One implication follows from the continuity of f. For the other, assume $U \subseteq Y$ is such that $f^{-1}(U) \subseteq X$ is open. Then $X \setminus f^{-1}(U)$ is closed, and hence by (1) $f(X) \setminus U = f(X \setminus f^{-1}(U))$ is also closed. Since f is surjective, $X \setminus U$ is closed, therefore $U \subseteq Y$ is open as needed. (3) For a bijection $f\colon X \to Y$, the map f is closed (or open) if and only if the (set-theoretically defined) map $f^{-1}\colon Y \to X$ is continuous. □

1.90. **Lemma** *Let X be a compact Hausdorff space and $x \in X$. The connected component $C(x)$ of x is the intersection of all open and closed subsets of X containing x.*

Proof. Exercise. □

1.6. **Mapping spaces.** Let X, Y be topological spaces. We wish to endow the set $C(X,Y)$ of continuous maps from X to Y with a topology. There are many topologies one could consider, the one most suitable for us is the following one.

1.91. **Definition** Let X, Y be topological spaces. For $K \subseteq X$ and $U \subseteq Y$, define $\mathrm{O}_{K,U} = \{f \in C(X,Y) \mid f(K) \subseteq U\}$. Then the set

$$\{\mathrm{O}_{K,U} \mid K \subseteq X \text{ compact and } U \subseteq Y \text{ open}\}$$

forms the subbasis of the *compact-open* topology on $C(X,Y)$. We denote the resulting topological space by $\mathrm{Map}(X,Y)$.

1.92. **Lemma** *Let $\mathcal{S}$ be a subbasis for the topology on Y and X be a Hausdorff space. Then the set*

$$\{\mathrm{O}_{K,U} \mid K \subseteq X \textit{ compact and } U \subseteq \mathcal{S}\}$$

is a subbasis for the compact-open topology on $\mathrm{Map}(X,Y)$.

Proof. Let $U \subseteq Y$ open and $K \subseteq X$ compact and let $f \in \mathrm{O}_{K,U}$. Write $U = \bigcup_{i \in I} U_i$ where $U_i = \bigcap_{j=1}^{m_i} A_{i_j}$ with $A_{i_j} \in \mathcal{S}$. Then $K \subseteq \bigcup_{i \in I} f^{-1}(U_i)$. For $x \in K$, pick $i(x) \in I$ such that $x \in f^{-1}(U_i)$ and pick $x \in V_x \subseteq K_x \subseteq K \cap f^{-1}(U_{i(x)})$ with V_x open and K_x compact. This can be done since K is compact Hausdorff and hence locally compact by Corollary 1.88. Since K

is compact, there are $x_1,\dots,x_n \in K$ such that $K \subseteq \bigcup_{k=1}^n V_{x_k}$. We note that $f \in \mathrm{O}_{K_{x_k},U_{i(x_k)}}$ for all $k = 1,\dots,n$. Recall now that $U_{i(x_k)} = \bigcap_{j=1}^{m_k} A_{i_j}$ with $A_{i_j} \in \mathcal{S}$. Then we have

$$f \in \bigcap_{k=1}^{n} \bigcap_{j=1}^{m_k} \mathrm{O}_{K_{x_k},A_{i_j}} = \bigcap_{k=1}^{n} \mathrm{O}_{K_{x_k},U_{i(x_k)}} \subseteq \mathrm{O}_{K,U}$$

showing that $\mathrm{O}_{K,U}$ is open in the topology generated by the $\mathrm{O}_{K,V}$'s with $V \in \mathcal{S}$. □

1.93. **Lemma** *Let $f\colon X' \to X$ and $g\colon Y \to Y'$ be continuous map. Then the maps* $\mathrm{Map}(X,Y) \xrightarrow{f^*} \mathrm{Map}(X',Y)$ *and* $\mathrm{Map}(X,Y) \xrightarrow{g_*} \mathrm{Map}(X,Y')$ *are continuous.*

Proof. We have $(f^*)^{-1}(\mathrm{O}_{K,U}) = \mathrm{O}_{f(K),U}$ and $(g_*)^{-1}(\mathrm{O}_{K,U}) = \mathrm{O}_{K,g^{-1}(U)}$. The lemma then follows from the fact that $f(K)$ is compact if K is and that $g^{-1}(U)$ is open if U is. □

1.94. **Corollary** *The association* $(X,Y) \mapsto \mathrm{Map}(X,Y)$ *refines to a functor* $\mathrm{Top}^{\mathrm{op}} \times \mathrm{Top} \to \mathrm{Top}$.

1.95. **Example** Let X be a topological space. Then $\mathrm{Map}(*,X)$ is homeomorphic to X via the evaluation map. More, generally, for any set I, we have $\mathrm{Map}(I^\delta,X)$ is homeomorphic to $\prod_I X$ again via the evaluation maps. Here, I^δ denotes I equipped with the discrete topology. Indeed, in both cases the maps are clearly bijections, and it is elementary to check that both sides have the same subbasis for the respective topologies.

1.96. **Notation** Let X be a topological space. We write $L(X)$ for $\mathrm{Map}(S^1,X)$ for the *free loop space* in X. Given a point $x \in X$, we denote by $\Omega_x(X) = \mathrm{Map}_*(S^1,X)$, the *based loop space*, i.e. the subspace of $\mathrm{Map}(S^1,X)$ consisting of those maps $\gamma\colon S^1 \to X$ sending $1 \in S^1$ to x, see Definition 1.105. More generally, for $x,y \in X$, we denote by $\Omega_{x,y}(X)$ the pullback

$$\begin{array}{ccc} \Omega_{x,y}(X) & \longrightarrow & \mathrm{Map}([0,1],X) \\ \downarrow & & \downarrow \\ \{(x,y)\} & \longrightarrow & X \times X \end{array}$$

Exercise: Show that $\Omega_{x,x}(X)$ is homeomorphic to $\Omega_x(X)$. Hint: Show that $[0,1]/0 \sim 1$ is homeomorphic to S^1.

1.97. **Remark** Based loop spaces play an important role in homotopy theory, as they are topological versions of groups (under concatenation of based loops). We will come to some aspects of this later when discussing homotopy groups of spaces. Free loop spaces (in particular of manifolds) play a fundamental role in differential topology and mathematical physics. The algebraic structure obtained from free loop spaces (via e.g. homology – an invariant we will introduce at in the second half of this course) even has its own name: It is called *string topology*.

1.98. **Remark** Let $f\colon (X,x) \to (Y,y)$ be a pointed map. Then f induces a canonical pointed map $\Omega_x(X) \to \Omega_y(Y)$. This map gives the association $(X,x) \to \Omega_x(X)$ the structure of a functor $\mathrm{Top}_* \to \mathrm{Top}_*$.

We come to another important structure the set of continuous maps has: For spaces X, Y and Z, there is a well-defined composition map:

$$C(Y,Z) \times C(X,Y) \longrightarrow C(X,Z), \quad (g,f) \mapsto g \circ f.$$

One may wonder whether this map is continuous with respect to the compact open topologies on the set of continuous maps, and the product topology on the domain of the above map. It is a somewhat annoying fact that this is not the case, however it is so under mild assumptions, often satisfied in practice.

1.99. **Proposition** *Let X, Y and Z be topological spaces and let Y be locally compact. Then the map*

$$\mathrm{Map}(Y,Z) \times \mathrm{Map}(X,Y) \xrightarrow{\circ} \mathrm{Map}(X,Z), \quad (g,f) \mapsto g \circ f$$

is continuous.

Proof. Pick $K \subseteq X$ compact and $U \subseteq Z$ open, and (g,f) such that $gf \in \mathrm{O}_{K,U}$, i.e. such that $f(K) \subseteq g^{-1}(U)$. Since Y is locally compact, for each $y \in f(K)$, we can find $K_y \subseteq g^{-1}(U)$ compact and $U_y \subseteq g^{-1}(U)$ open such that $y \in U_y \subseteq K_y$. Then $\bigcup_{y \in f(K)} U_y \supseteq f(K)$, so by compactness of $f(K)$ we find $y_1, \dots, y_n \in f(K)$ such that $U_i := U_{y_i}$ satisfy $f(K) \subseteq V = U_1 \cup \dots \cup U_n$. Let $L = K_1 \cup \dots \cup K_n$ which is a compact subset of Y and $V \subseteq L$. Then we find that $\mathrm{O}_{L,U} \times \mathrm{O}_{K,V} \subseteq \circ^{-1}(\mathrm{O}_{K,U})$ as needed. □

1.100. **Remark** A special case of the above is when $X = *$. Under the homeomorphisms $\mathrm{Map}(*,X) \cong X$ and $\mathrm{Map}(*,Z) \cong Z$ of Example 1.95, the composition map becomes the evaluation map $\mathrm{ev}\colon \mathrm{Map}(Y,Z) \times Y \to Z$. It follows that this map is continuous when Y is locally compact.

Finally, we discuss the relationship between the functors $X \times -$ and $\mathrm{Map}(X,-)$. Recall that in the category of sets, we have that $A \times -$ is left adjoint to $\mathrm{Hom}_{\mathrm{Set}}(A,-)$. We would like to have the same be true in topological spaces. However, again this is true only under additional assumptions. A convenient one is the following.

1.101. **Proposition** *Let X, Y and Z be topological spaces. Then the map*

$$c\colon \mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y,Z)), \quad f \mapsto \hat{f}\colon (x \to f(x,-))$$

is well-defined. In addition it is

(1) *continuous if X is Hausdorff,*
(2) *bijective if Y is locally compact, and*
(3) *a homeomorphis if X and Y are locally compact.*

Proof. To see well-definedness, we need to show that if $f\colon X \times Y \to Z$ is continuous, then so is $\hat{f}$. Let $U \subseteq Z$ be open and $K \subseteq Y$ be compact. Let $x \in \hat{f}^{-1}(\mathrm{O}_{K,U})$ and consider $f^{-1}(U) \subseteq X \times Y$. By assumption $\{x\} \times K \subseteq f^{-1}(U)$. Since products of open sets form a basis of the topology on $X \times Y$, for each $k \in K$ we find opens $V_k \subseteq X$ containing x and $W_k \subseteq Y$ containing k such that $U_k \times V_k \subseteq f^{-1}(U)$. Since K is compact, there are $k_1, \dots, k_n$ such that $K \subseteq W_1 \cup \dots \cup W_n$. Let $V = V_1 \cap \dots \cap V_n$, which is open in X. Then $V \times K \subseteq f^{-1}(U)$, and therefore $V \subseteq \hat{f}^{-1}(\mathrm{O}_{K,U})$. The self-inedxing trick shows that $\hat{f}^{-1}(\mathrm{O}_{K,U})$ is open and so $\hat{f}$ is continuous. (1) Since X is Hausdorff, we find that $\mathrm{Map}(X, \mathrm{Map}(Y,X))$ has a subbasis consisting of $\mathrm{O}_{K,\mathrm{O}_{L,U}}$ where $K \subseteq X$ compact, $L \subseteq Y$ compact and $U \subseteq Z$ open. Then $c^{-1}(\mathrm{O}_{K,\mathrm{O}_{L,U}}) = \mathrm{O}_{K \times L,U}$ which is open in $\mathrm{Map}(X \times Y, Z)$ since $K \times L$ is compact by

Tychonoff's theorem. (2) To see that the map c is bijective if Y is locally compact, we need to show that f is continuous if $\hat{f}$ is. To see the this, note that f is given by the composite

$$X \times Y \xrightarrow{\hat{f}\times \mathrm{id}_Y} \mathrm{Map}(Y, Z) \times Y \xrightarrow{\mathrm{ev}} Z.$$

This composite is continuous since each individual map appearing in it is continuous. Indeed, the evaluation map is continuous by the assumption that Y is locally compact, see Remark 1.100. (3) To see that the map c is a homeomorphism if X and Y are locally compact, we first show that c is continuous. By the previously established results (applied twice), this is equivalent to the adjoint map

$$\mathrm{Map}(X \times Y, Z) \times X \times Y \to Z$$

being continuous. This map is the evaluation map and hence continuous since X and Y, and therefore by Tychonoff, also $X \times Y$ is locally compact, see Addendum 1.78. It remains to show that the inverse of c

$$\mathrm{Map}(X, \mathrm{Map}(Y, Z)) \to \mathrm{Map}(X \times Y, Z)$$

is also continuous. Again, we use the previously established results and the fact that $X \times Y$ is continuous, to see that it suffices to show that the adjoint map

$$\mathrm{Map}(X, \mathrm{Map}(Y, Z)) \times X \times Y \to Z$$

is continuous. This map factors as the composite

$$\mathrm{Map}(X, \mathrm{Map}(Y, Z)) \times X \times Y \to \mathrm{Map}(Y, Z) \times Y \to Z$$

where the first map is the evaluation map for X times the identity on Y and the second map is the evaluation map for Y. Both of these maps are continuous since X and Y are locally compact. □

1.102. **Remark** In fact, $\mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z))$ is a homeomorphism if X is Hausdorff and Y is locally compact Hausdorff. We leave this as an exercise for the interested reader.

1.103. **Corollary** *Let Y be a locally compact space. Then $Y \times -$ is left adjoint to* $\mathrm{Map}(Y, -)$.

Proof. By Proposition 1.101, the canonical map

$$\mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

is a bijection when Y is locally compact. The underlying set of the mapping spaces are just the continuous maps, so the corollary follows. □

1.104. **Corollary** *Let Y be a locally compact space. Then the functor $Y \times -$ preserves colimits. In particular, it preserves quotient maps.*

We have observed earlier that pointed spaces also come with the smash product of spaces. There is then also the pointed version of the mapping space:

1.105. **Definition** For pointed spaces (X, x) and (Y, y) we denote by $\mathrm{Map}_*((X, x), (Y, y))$ the subspace of $\mathrm{Map}(X, Y)$ on pointed continuous maps.

As a consequence of Lemma 1.93, we also find that $\mathrm{Map}_*(-,-)\colon \mathrm{Top}_*^{\mathrm{op}} \times \mathrm{Top}_* \to \mathrm{Top}_*$ is a functor (all maps are simply restricted from the unpointed case). Moreover, we find that the composition of unpointed mapping spaces restricts to the composition of pointed mapping spaces

$$\mathrm{Map}_*(Y,Z) \times \mathrm{Map}_*(X,Y) \to \mathrm{Map}_*(X,Z)$$

and is therefore continuous when the underlying unpointed space of Y is locally compact. As a result of these arguments, we find the following.

1.106. **Corollary** *Let X, Y and Z be pointed topological spaces. Then the map*

$$\mathrm{Map}_*(X \wedge Y, Z) \to \mathrm{Map}_*(X, \mathrm{Map}_*(Y,Z)), \quad f \mapsto \hat{f}\colon (x \to f(x \wedge -))$$

is well-defined and continuous if X is Hausdorff. It is bijective if Y is locally compact and a homeomorphism if X and Y are locally compact. In particular, the functor $Y \wedge -$ is a left adjoint whenever the underlying space of Y is locally compact.

Finally, we will use the following basic result.

1.107. **Lemma** *Let X, X_i, Y, and Y_i be families of topological spaces indexed over a set I. Then the canonical maps*

$$\mathrm{Map}(\coprod_{i \in I} X_i, Y) \to \prod_{i \in I} \mathrm{Map}(X_i, Y)$$

as well as

$$\mathrm{Map}(X, \prod_{i \in I} Y_i) \to \prod_{i \in i} \mathrm{Map}(X, Y_i)$$

are continuous bijections. The first one is a homeomorphism and the second one is a homeomorphism if X is locally compact.

Proof. The maps are continuous, simply because $\mathrm{Map}(X,-)$ and $\mathrm{Map}(-Y)$ are functors. The maps are bijections by the universal properties of coproducts and products, respectively. It remains to see that the inverses are continuous. So let $K \subseteq \coprod_{i \in i} X_i$ be compact. Then $K_i = K \cap X_i$ is non-empty in at most finitely many cases (since K is compact). For $U \subseteq Y$ open, consider $\mathrm{O}_{K,U}$. The image oder this map above is then given by the product of the sets $\mathrm{O}_{K_i,U}$ of which only finitely many are not all of $\mathrm{Map}(X_i, Y)$ and all are open. This is an open subset for the product topology, so the first claim is shown. To see the second case, we wish to show that the canonical bijective map

$$\prod_{i \in I} \mathrm{Map}(X, Y_i) \to \mathrm{Map}(X, \prod_{i \in I} Y_i)$$

is continuous. Since X is locally compact this is the case if and only if its adjoint map

$$\prod_{i \in I} \mathrm{Map}(X, Y_i) \times X \to \prod_{i \in I} Y_i$$

is continuous. By the universal property of the product, this map is continuous if and only if it is so after postcomposition with the projections. Such a composite is readily seen to be given by the composite

$$\prod_{i \in I} \mathrm{Map}(X, Y_i) \times X \to \mathrm{Map}(X, Y_j) \times X \to Y_j$$

which is continuous as a composition of continuous maps, we use again that X is locally compact for the latter map to be continuous. □

1.108. **Remark** Again, the similar result is true in the pointed case: Here we find that the maps

$$\mathrm{Map}_*(\bigvee_{i\in I} X_i, Y) \to \prod_{i\in I} \mathrm{Map}_*(X_i, Y)$$

as well as

$$\mathrm{Map}_*(X, \prod_{i\in I} Y_i) \to \prod_{i\in i} \mathrm{Map}_*(X, Y_i)$$

are continuous bijections. The first one is a homeomorphism if I is finite and the second one is a homeomorphism if X is locally compact.

1.109. **Example** Recall that the forgetful functor $\mathrm{Top}_* \to \mathrm{Top}$ has a left adjoint given by sending X to $X_+ = X \amalg \{*\}$. We conclude that there is a canonical homeomorphism $\mathrm{Map}_*(X_+, Y) \cong \mathrm{Map}(X, Y)$. Therefore, statements about unpointed mapping spaces can often be reduced to statements about pointed mapping spaces.

2. A primer on homotopy theory

2.1. Basic definitions.

We begin with the basic definitions of homotopy theory.

2.1. **Definition** Let $A \subseteq X$ and Y be topological spaces. Fix a map $\varphi\colon A \to Y$ and denote by $C_A(X,Y) \subseteq C(X,Y)$ the subset of continuous maps $X \to Y$ whose restriction to A is given by φ. We say that two such maps f and g are *homotopic rel* A, if there exists a map $H\colon [0,1] \times X \to Y$ such that

(1) $H(t,a) = \varphi(a)$ for all $t \in [0,1]$,
(2) $H(0,-) = f$, and
(3) $H(1,-) = g$.

A map H as above is called a *homotopy rel* A. An important special case is when $A = \varnothing$, and when $A = \{x\}$ consists of a basepoint. In this case we say based or pointed homotopy rather than homotopy rel $\{x\}$.

2.2. **Lemma** *The relation "homotopy rel A" is an equivalence relation on* $C_A(X,Y)$.

Proof. Exercise. □

We write $[X,Y]_A$ for the set of rel A homotopy classes of maps $X \to Y$. In case $A = \varnothing, *$, we write $[X,Y]$ and $[X,Y]_*$.

2.3. **Remark** In the situation of Definition 2.1, suppose that X is locally compact and denote by $\mathrm{Map}_A(X,Y)$ the subspace of $\mathrm{Map}(X,Y)$ on the elements of $C_A(X,Y)$. Then a homotopy rel A is the same datum as a continuous map $[0,1] \to \mathrm{Map}_A(X,Y)$. In particular, the equivalence classes of the homotopy rel A relation $[X,Y]_A$ is bijective to $\pi_0(\mathrm{Map}_A(X,Y))$. If X is not locally compact, this need not be the case.

Indeed, since X is assumed to be locally compact, a homotopy $H\colon [0,1] \times X \to Y$ is equivalently given by a continuous map $\hat{H}\colon [0,1] \to \mathrm{Map}(X,Y)$. The three conditions in the definition of a homotopy rel A translate the following properties:

(1) $\hat{H}(t) \in \mathrm{Map}_A(X,Y)$,
(2) $\hat{H}(0) = f$, and
(3) $\hat{H}(1) = g$

as claimed.

2.4. **Definition** Let $f\colon X \to Y$ be a continuous map. f is called a *homotopy equivalence* if there exists $g\colon Y \to X$ continuous and homotopies H_1 from fg to id_Y and H_2 from gf to id_X. Such a map g is called a homotopy inverse of f. Two spaces X and Y are called *homotopy equivalent* if there exists a homotopy equivalence $f\colon X \to Y$. A space X is called contractible if it is homotopy equivalent to $*$.

2.5. **Remark** Likewise, one defines the notion of a pointed homotopy equivalence. This is a continuous pointed map $f\colon (X,x) \to (Y,y)$ such that there exists a pointed map $g\colon (Y,y) \to (X,x)$ and pointed homotopies $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$. Two pointed homotopy equivalent spaces are also homotopy equivalent as unpointed spaces. Then converse is not true in general, see Example 2.13 below, but it does hold under additional assumptions on the basepoints (namely that $\{x\} \to X$ and $\{y\} \to Y$ are cofibrations, a notion we will introduce soon).

2.6. **Remark** Unlike for homeomorphisms, a homotopy inverse (if it exists) need not be unique. However: (1) Let $f\colon X \to Y$ be a homotopy equivalence and g, g' be homotopy inverses. Then g and g' are homotopic as we will show next. (2) A space X is contractible if and only if it is non-empty, and id_X is homotopic to the constant map at any given basepoint of X.

2.7. **Lemma** *Let $f, f'\colon X \to Y$ and $g, g'\colon Y \to Z$ and $A \subseteq X$ with $f_{|A} = f'_{|A}$ and $B \subseteq Y$ with $g_{|B} = g'_{|B}$. If f and f' are homotopic rel A, then so are gf and gf'. Likewise, if g and g' are homotopic rel B, then gf and $g'f$ are homotopic rel $f^{-1}(B)$. In particular,*

(1) *If f and f' are homotopic and g and g' are homotopic, then gf and $g'f'$ are homotopic.*
(2) *If f and f' are pointed homotopic and g and g' are pointed homotopic, then gf and $g'f'$ are pointed homotopic.*

if both are true, gf and $g'f'$ are homotopic.

Proof. Pick H a homotopy rel A from f to f'. Then the map $gH\colon [0,1] \times X \to Z$ is a homotopy rel A as required. Likewise pick H' a homotopy rel B from g to g'. Then the composite

$$[0,1] \times X \xrightarrow{\mathrm{id}\times f} [0,1] \times Y \xrightarrow{H'} Z$$

is a homotopy rel $f^{-1}(B)$ from gf to $g'f$. (1) is then the special case where $A = B = \varnothing$. (2) follows from the special case where $A = \{x\}$ and $B = \{y\}$. □

2.8. **Remark** The above implies that one can define categories hTop and hTop_*, the *homotopy category of (pointed) topological spaces* whose objects are (pointed) topological spaces and whose morphisms are given by (pointed) homotopy classes of (pointed) maps. Lemma 2.7 implies that defining composition on representatives of homotopy classes is well-defined. There are canonical functors $\mathrm{Top}_{(*)} \to \mathrm{hTop}_{(*)}$ which are the identity on objects and the canonical projection on morphism sets. Exercise: A functor $\mathrm{hTop}_{(*)} \to \mathcal{C}$ is equivalently described by a functor $\mathrm{Top}_{(*)} \to \mathcal{C}$ having the *property* that (pointed) homotopic maps are sent to equal maps. Such functors are henceforth called *(pointed) homotopy invariant* functors. Indeed, more is true: the functor $\mathrm{Fun}(\mathrm{hTop}_{(*)}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Top}_{(*)}, \mathcal{C})$ is fully faithful with essential image the (pointed) homotopy invariant functors.

Note that a (pointed) homotopy invariant functor sends (pointed) homotopy equivalences to isomorphisms.

2.9. **Example** The functor $\pi_0\colon \mathrm{Top} \to \mathrm{Set}$ is homotopy invariant. Indeed, we first observe that the two inclusion $\{0\} \to [0,1]$ and $\{1\} \to [0,1]$ induce the same map on π_0 since $\pi_0([0,1])$ consists of a single point. Now suppose $f, g\colon X \to Y$ are homotopic and pick a homotopy H from f to g. Then H induces a map $\pi_0(X \times [0,1]) \to \pi_0(Y)$. Since π_0 commutes with products, this is equivalently a map $\pi_0(H)\colon \pi_0(X) \times \pi_0([0,1]) \to \pi_0(Y)$. Restriction along $\pi_0(X) \cong \pi_0(X) \times \pi_0(\{i\})$ gives $\pi_0(f)$ for $i = 0$ and $\pi_0(g)$ for $i = 1$, so the claim follows from the previously established fact.

2.10. **Lemma** *Let $f, f'\colon X \to X'$ be (pointed) homotopic maps and let Y be a further (pointed) topological space. Then the induced maps*

$$\mathrm{Map}_{(*)}(X', Y) \xrightarrow{f^*, f'^*} \mathrm{Map}_{(*)}(X, Y) \quad \textit{and} \quad \mathrm{Map}_{(*)}(Y, X) \xrightarrow{f_*, f'_*} \mathrm{Map}_{(*)}(Y, X')$$

are (pointed) homotopic. Likewise, the maps

$$Y \times X \xrightarrow{Y\times f, Y\times f'} Y \times X' \quad \textit{and} \quad Y \wedge X \xrightarrow{Y\wedge f, Y\wedge f'} Y \wedge X'$$

are (pointed) homotopic.

Proof. Pick a homotopy $H\colon X \times [0,1] \to X'$ from f to f' and consider the map

$$\mathrm{Map}(X', Y)\times[0,1] \xrightarrow{H^*} \mathrm{Map}(X\times[0,1], Y)\times[0,1] \to \mathrm{Map}([0,1], \mathrm{Map}(X, Y))\times[0,1] \to \mathrm{Map}(X, Y)$$

of which the second map is continuous by Proposition 1.101 and the latter is continuous by Remark 1.100. It is readily checked to be a homotopy between f^* and f'^*. Likewise, consider the map

$$\mathrm{Map}(Y, X) \times [0,1] \to \mathrm{Map}(Y, X \times [0,1]) \xrightarrow{H_*} \mathrm{Map}(Y, X')$$

of which the first map sends (f, t) to the map $y \mapsto (f(y), t)$. This map is continuous (see Exercise below). The above composite is again readily checked to be a homotopy from f_* to f'_*. If H is a pointed homotopy, simply observe that the above two composites restrict to (consequently continuous) pointed homotopies

$$\mathrm{Map}_*(X', Y) \times [0,1] \to \mathrm{Map}_*(X, Y) \quad \text{and} \quad \mathrm{Map}_*(Y, X) \times [0,1] \to \mathrm{Map}_*(Y, X').$$

The case of products and smash products follows from considering $\mathrm{id}_X \times H\colon Y \times X \times [0,1] \to Y \times X'$ and using that this map descends to a continuous map on smash products in case H is a pointed homotopy. □

Exercise. Let X, Y and Z be topological spaces. Then the map

$$\mathrm{Map}(X, Y) \times Z \to \mathrm{Map}(X, Y \times Z), \quad (f, z) \mapsto (x \mapsto (f(x), z))$$

is continuous.

2.11. **Definition** Let $A \subseteq X$ be topological spaces. We say A is a deformation retraction of X if there exists a map $\varphi \in C_A(X, A)$ and a homotopy rel A from φ to id_A.

2.12. **Example**

(1) $\{0\}$ is a deformation retraction of $\mathbb{R}^n$. Indeed, the map $[0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ given by $(t, x) \mapsto tx$ provides a homotopy rel $\{0\}$ from the map constant at 0 to the identity of $\mathbb{R}^n$.

(2) $S^{n-1} \subseteq \mathbb{R}^n \setminus \{0\}$ is a deformation retraction. Indeed, the map $x \mapsto \frac{x}{\|x\|}$ is a continuous map $\mathbb{R}^n \setminus \{0\} \to S^{n-1}$ and the map $[0,1] \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ given by $(t,x) \mapsto \frac{x}{t+(1-t)\|x\|}$ provides a homotopy rel S^{n-1} as needed.

2.13. **Example** A deformation retraction $A \subseteq X$ is a homotopy equivalence. There are examples of subspace inclusions $A \subseteq X$ which are homotopy equivalences, but not deformation retractions. For instance, let us examine the special case $A = \{x\} \subseteq X$. Then $\{x\} \to X$ being a homotopy equivalence means that X is contractible and that $\{x\}$ is a deformation retraction simply means that X is pointed contractible (i.e. that there is a pointed homotopy from the identity of X to the constant map at x). Without proof, we mention here that the space

$$S = \{(x,y) \subseteq \mathbb{R}^2 \mid y = xm \text{ for some } m \in \mathbb{Q}\}$$

is contractible (the map $H(t,(x,y)) = (tx,ty)$ is a homotopy between the identity and the constant map at $(0,0)$), but S does not deformation retract to the point $(1,0)$.

We come to the following fundamental geometric structure. I thank Tyrone Cutler (and his online available notes) for having written down the explicit formulas we use here.

2.14. **Lemma** *The pointed space* $(S^1,1)$ *is canonically endowed with the structure of a comonoid up to homotopy in* Top_*[1]. *That is, there are the following maps:*

(1) *the counit map* $t\colon S^1 \to *$, *and*
(2) *the comultiplication map* $p\colon S^1 \to S^1 \vee S^1$.

These satisfy counitality and coassciativity up to homotopy, that is, the following diagrams commute up to homotopy:

$$\begin{array}{ccc} S^1 & \xrightarrow{p} & S^1 \vee S^1 \\ & \searrow^{\mathrm{id}} & \downarrow t\vee\mathrm{id} \ \downarrow \mathrm{id}\vee t \\ & & S^1 \end{array} \qquad \begin{array}{ccc} S^1 & \xrightarrow{p} & S^1 \vee S^1 \\ \downarrow p & & \downarrow \mathrm{id}\vee p \\ S^1 \vee S^1 & \xrightarrow{p\vee\mathrm{id}} & S^1 \vee S^1 \vee S^1 \end{array}$$

Moreover, the so defined comonoid $(S^1,1)$ *is in fact a cogroup. That it is, the following equivalent conditions are satisfied:*

(1) *The map* $p \vee \iota_r\colon S^1 \vee S^1 \to S^1 \vee S^1$ *is a pointed homotopy equivalence,*
(2) *There exists a pointed map* $\mathrm{coinv}\colon S^1 \to S^1$, *called the coinversion map, such that the composites*

$$S^1 \xrightarrow{p} S^1 \vee S^1 \underset{\mathrm{id}\vee\mathrm{coinv}}{\overset{\mathrm{coinv}\vee\mathrm{id}}{\rightrightarrows}} S^1 \vee S^1 \xrightarrow{\nabla} S^1$$

are pointed homotopic to the constant map. Here, ∇ *is the fold map, i.e. the map which is the identity on each of the two wedge summands.*

Proof. First, we recall that the map $t \mapsto \exp(t) = e^{2\pi i t}$ induces a homeomorphism $[0,1]/0 \sim 1 \to S^1$. We will from now on identity S^1 with $[0,1]/0 \sim 1$ via this map without further notation. We define the pinch map $p\colon S^1 \to S^1 \vee S^1$ to be induced by

$$[0,1] \ni t \mapsto \begin{cases} \iota_r(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \iota_l(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

[1] Equivalently, the image of $(S^1,1)$ under the functor $\mathrm{Top}_* \to \mathrm{hTop}_*$ is canonically endowed with a comonoid structure.

Here, ι_r and ι_l are the canonical right and left inclusions of S^1 into $S^1 \vee S^1$. Geometrically, p is the quotient map $S^1 \to S^1/\{\pm 1\}$ obtained by collapsing $S^0 \subseteq S^1$ to a point.

Let us now consider the above diagrams. First, we consider the composite $(\mathrm{id} \vee t)p\colon S^1 \to S^1$. It is induced by the map

$$[0,1] \ni t \mapsto \begin{cases} 2t & \text{for } 0 \leq t \leq \frac{1}{2} \\ * & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

where $* = [0]$ is the basepoint of S^1. The map $H\colon [0,1] \times [0,1] \to S^1$ given by

$$H(s,t) = \begin{cases} \frac{2t}{1+s} & \text{for } 0 \leq t \leq \frac{1+s}{2} \\ * & \text{for } \frac{1+s}{2} \leq t \leq 1 \end{cases}$$

induces the desired pointed homotopy from $(\mathrm{id} \vee t)p$ to id_{S^1}. That $(t \vee \mathrm{id})p$ is pointed homotopic to id_{S^1} is a similar argument.

Now we work out the second diagram. We denote by ι_m the middle inclusion of S^1 into $S^1 \vee S^1 \vee S^1$. With this in mind, we have

$$(\mathrm{id} \vee p)p(t) = \begin{cases} \iota_l(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \iota_m(4t-2) & \text{for } \frac{1}{2} \leq t \leq \frac{3}{4} \\ \iota_r(4t-3) & \text{for } \frac{3}{4} \leq t \leq 1 \end{cases}, \; (p \vee \mathrm{id})p(t) = \begin{cases} \iota_l(4t) & \text{for } 0 \leq t \leq \frac{1}{4} \\ \iota_m(4t-1) & \text{for } \frac{1}{4} \leq t \leq \frac{1}{2} \\ \iota_r(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

We claim that both these maps are pointed homotopic to

$$t \mapsto \varphi(t) = \begin{cases} \iota_l(3t) & \text{for } 0 \leq t \leq \frac{1}{3} \\ \iota_m(3t-1) & \text{for } \frac{1}{3} \leq t \leq \frac{2}{3} \\ \iota_r(3t-2) & \text{for } \frac{2}{3} \leq t \leq 1 \end{cases}$$

For this, we consider the map

$$[0,1] \times [0,1] \ni (t,s) \mapsto \begin{cases} \iota_l((4-s)t) & \text{for } 0 \leq t \leq \frac{1}{4-s} \\ \iota_m(\frac{(4-s)(2+s)t-(2+s)}{(4-s)(1+s)-(2+s)}). & \text{for } \frac{1}{4-s} \leq t \leq \frac{1+s}{2+s} \\ \iota_r((2+s)t-(1+s)) & \text{for } \frac{1+s}{2+s} \leq t \leq 1 \end{cases}$$

which induces a pointed homotopy from $(p \vee \mathrm{id})p$ to φ.

That the conditions (a) and (b) are equivalent appears on Exercise Sheet 5. We show that condition (b) is satisfied. To do so, we define coinv to be induced by the map $[0,1] \to [0,1]$ which sends t to $(1-t)$. Geometrically, this is the the loop on S^1 which runs around S^1 once clockwise. Then the composites $\nabla(\mathrm{coinv} \vee \mathrm{id})p$ and $\nabla(\mathrm{id} \vee \mathrm{coinv})p$ are induced by the maps

$$[0,1] \ni t \mapsto \begin{cases} 1-2t & \text{for } 0 \leq t \leq \frac{1}{2} \\ 2t-1 & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}, \text{ and } \quad t \mapsto \begin{cases} 2t & \text{for } 0 \leq t \leq \frac{1}{2} \\ 2-2t & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

We need to show that both are homotopic to the constant map at the basepoint. To do so, we consider the map

$$[0,1] \times [0,1] \ni (t,s) \mapsto \begin{cases} 2t & \text{for } 0 \leq t \leq \frac{1-s}{2} \\ 1-s & \text{for } \frac{1-s}{2} \leq t \leq \frac{1+s}{2} \\ 2-2t & \text{for } \frac{1+s}{2} \leq t \leq 1 \end{cases}$$

which provides a pointed homotopy from $\nabla(\mathrm{id} \vee \mathrm{coinv})p$ to the constant path. A similar homotopy exists for $\nabla(\mathrm{coinv} \vee \mathrm{id})p$. □

2.15. **Remark** It turns out (we might come back to this later) that the cogroup structure on $(S^1, 1) \in \mathrm{hTop}_*$ is unique. Moreover, it does not come from a strict cogroup structure on S^1, i.e. $(S^1, 1) \in \mathrm{Top}_*$ does *not* admit the structure of a cogroup. This means that a) there was really no choice other than the pinch map for the comultiplication, and b) we really have to introduce some non-trivial homotopies which witness the counitality and coassociativity of the comultipliaction up to homotopy.

2.16. **Corollary** *Let (X, x) be a pointed space. Then $S^1 \wedge X$ is a cogroup up to homotopy in* Top_*.

Proof. For all pointed spaces X, Y and Z, there is a canonical homeomorphism $(X \vee Y) \wedge Z \cong (X \vee Z) \wedge (Y \vee Z)$ (this is not completely trivial to show, but we will not give the details here). Consequently, the cogroup structure on S^1 defines maps

$$S^1 \wedge X \longrightarrow (S^1 \vee S^1) \wedge X \cong (S^1 \wedge X) \vee (S^1 \wedge X), \quad S^1 \wedge X \to * \wedge X \cong *$$

which satisfy counitality and coassociativity up to homotopy since the functor $-\wedge X$ preserves pointed homotopies, see Lemma 2.10. Hence $S^1 \wedge X$ is a comonoid up to homotopy. To see that this is a cogroup, it suffices to see that $\mathrm{coinv} \wedge X \colon S^1 \wedge X \to S^1 \wedge X$ is a coinversion map for the comonoid structure just defined. Again, this follows from the fact that $- \wedge X$ preserves pointed homotopies. □

Similarly, we have the following:

2.17. **Corollary** *Let (X, x) be a pointed space. Then $\Omega_x(X)$ is a group up to homotopy in* Top_*.

Proof. We recall that $\Omega_x(X) = \mathrm{Map}_*(S^1, X)$. The cogroup structure on S^1 induces the following maps

$$\mathrm{Map}_*(S^1, X) \times \mathrm{Map}_*(S^1, X) \cong \mathrm{Map}_*(S^1 \vee S^1, X) \longrightarrow \mathrm{Map}_*(S^1, X)$$

as well as $* = \mathrm{Map}_*(*, X) \to \mathrm{Map}_*(S^1, X)$. These maps satisfy unitality and associativity up to homotopy since the functor $\mathrm{Map}_*(-, X)$ preserves pointed homotopies, see Lemma 2.10. We have used Remark 1.108 for the above homeomorphism. Hence $\Omega_x(X)$ is a monoid up to homotopy. Moreover, the map $\mathrm{coinv}^* \colon \mathrm{Map}_*(S^1, X) \to \mathrm{Map}_*(S^1, X)$ provides an inversion map for $\Omega_x(X)$, so this is indeed a group up to homotopy. □

2.18. **Definition** Let X be a topological space. For any basepoint $x \in X$, we define the *fundamental group* of X at x, $\pi_1(X, x)$ as $\pi_0(\Omega_x(X))$.

2.19. **Lemma** *The association $(X, x) \mapsto \pi_1(X, x)$ refines to a pointed homotopy invariant functor* $\mathrm{Top}_* \to \mathrm{Grp}$.

Proof. If G is a group up to homotopy in Top, then $\pi_0(G)$ is an ordinary group: This follows from the fact that π_0 commutes with finite products and sends $*$ to $*$. Hence $\pi_1(X, x)$ is a group as a consequence of Corollary 2.17. Then we recall that a based map $f \colon X \to Y$ induces a map $\mathrm{Map}_*(S^1, X) \to \mathrm{Map}_*(S^1, Y)$, i.e. a map $\Omega_x(X) \to \Omega_x(Y)$. This map is one of groups up to homotopy since the diagram

$$\begin{array}{ccccc} \mathrm{Map}_*(S^1,X)\times\mathrm{Map}_*(S^1,X) & \xleftarrow{\cong} & \mathrm{Map}_*(S^1\vee S^1,X) & \longrightarrow & \mathrm{Map}_*(S^1,X) \\ \big\downarrow{\scriptstyle f_*\times f_*} & & \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle f_*} \\ \mathrm{Map}_*(S^1,Y)\times\mathrm{Map}_*(S^1,Y) & \xleftarrow{\cong} & \mathrm{Map}_*(S^1\vee S^1,Y) & \longrightarrow & \mathrm{Map}_*(S^1,Y) \end{array}$$

commutes (and similarly for the map defining the unit of the monoids $\mathrm{Map}_*(S^1,X)$ and $\mathrm{Map}_*(S^1,Y)$). Hence the induced map $\pi_1(X,x)\to\pi_1(Y,y)$ is a group homomorphism. Functoriality of these maps follows from the fact that $\pi_0(\mathrm{Map}_*(S^1,-))$ is a functor with values in Set and the fact that π_1 is pointed homotopy invariant follows from Lemma 2.10. □

We will need the following observation about the behaviour when changing the basepoint x via a path in X.

2.20. **Lemma** *Let $\gamma\colon[0,1]\to X$ be a path from x to x'. Then γ induces a pointed homotopy equivalence $\Omega_x(X)\to\Omega_{x'}(X)$, natural in X.*

Proof. We define a map $\Phi_\gamma\colon \mathrm{Map}_*((S^1,1),(X,x))\to\mathrm{Map}_*((S^1,1),(X,x'))$ by sending a loop α at x to the loop $\gamma\star\alpha\star\gamma^{-1}$ at x', that is, the concatenation of α with γ and its reverse γ^{-1}. The same argument that shows that concatentation with a fixed loop defines a continuous map $\Omega_x(X)\to\Omega_x(X)$ shows that the so defined map Φ_γ is continuous. For γ' a path from x' to x, the composite $\Phi_{\gamma'}\circ\Phi_\gamma$ is then homotopic to $\Phi_{\gamma'\star\gamma}$ - simply by reparametrising the path appropriately. Moreover, for any loop β at x, the map Φ_β is homotopic to the *conjugation by* β map of the group (up to homotopy) $\Omega_x(X)$. In particular, this map is a pointed homotopy equivalence: To see this, it suffices to show that the map is an isomorphism in hTop_*. But now in general, given a group G in a category $\mathcal{C}$, and an element $g\in G$, then conjugation by g defines an automorphism of G (with inverse given by conjugation by g^{-1}). It follows that Φ_γ is a homotopy equivalence with homotopy inverse given by $\Phi_{\gamma^{-1}}$.

Finally, to see the naturality of the construction, it suffices to note that for a continuous map $f\colon X\to Y$, the diagram

$$\begin{array}{ccc} \mathrm{Map}_*((S^1,1),(X,x)) & \xrightarrow{\Phi_\gamma} & \mathrm{Map}_*((S^1,1),(X,x')) \\ \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle f_*} \\ \mathrm{Map}_*((S^1,1),(Y,f(x))) & \xrightarrow{\Phi_{f(\gamma)}} & \mathrm{Map}_*((S^1,1),(Y,f(x'))) \end{array}$$

commutes strictly

$$f(\gamma\star\alpha\star\gamma^{-1})=f(\gamma)\star f(\alpha)\star f(\gamma^{-1})=f(\gamma)\star\alpha\star f(\gamma)^{-1}.$$

□

The notation π_1 and π_0 suggest that there might be variants π_n for any $n\geq 0$. In what follows, let $n\geq 1$ and consider the element $(1,0,\dots,0)\in\mathbb{R}^{n+1}$. For $k\leq n$ we have $\mathbb{R}^{k+1}\subseteq\mathbb{R}^{n+1}$ via the inclusion in the first $k+1$ coordinates. This gives a series of inclusions $S^0\subseteq S^1\subseteq\cdots\subseteq S^n$ and we consider 1 as an element of S^n via these inclusions.

2.21. **Remark** These inclusions can also be thought of as follows: We have $S^0\subseteq S^1$ as the subspace $\{\pm1\}$. Applying the functor $-\wedge S^1$ we obtain an inclusion $S^1\subseteq S^1\wedge S^1$ and inductively inclusions $S^1\wedge\cdots\wedge S^1\to S^1\wedge\cdots\wedge S^1\wedge S^1$. The claim then follows from a canonical homeomorphism $S^n \wedge S^m \cong S^{n+m}$. We will discuss this homeomorphism on Exercise Sheet 5 in light of the one-point compactification.

2.22. **Definition** Let (X, x) be a pointed space and $n \geq 0$. We define its nth homotopy group at x to be

$$\pi_n(X, x) = \pi_0(\mathrm{Map}_*((S^n, 1), (X, x)).$$

Strictly speaking, we have not seen that $\pi_n(X, x)$ is a group for $n \geq 2$. However, it follows from our earlier results:

2.23. **Corollary** *For all $n \geq 1$, the association $(X, x) \mapsto \pi_n(X, x)$ refines to a pointed homotopy invariant functor* $\mathrm{Top}_* \to \mathrm{Grp}$.

Proof. π_n is a homotopy invariant functor as the composite of the functors π_0 and $\mathrm{Map}_*((S^n, 1), -)$. To see that it canonically takes values in groups, we note that $S^n \cong S^k \wedge S^{n-k}$ and S^k and S^{n-k} are compact. This together with Corollary 1.106 implies that there are homeomorphisms

$$\mathrm{Map}_*((S^n, 1), (X, x)) \cong \mathrm{Map}_*((S^k, 1), \mathrm{Map}_*((S^{n-k}, 1), (X, x))).$$

In particular, we find for $n \geq 1$ bijections $\pi_n(X, x) \cong \pi_1(\mathrm{Map}_*((S^{n-1}, 1), (X, x))$. □

2.24. **Lemma** *Let (X, x) and (Y, y) be pointed spaces.*

(1) $\pi_0(X, x) = \pi_0(X)$
(2) *For all $n \geq 0$, the canonical map $\pi_n(X \times Y, (x, y)) \to \pi_n(X, x) \times \pi_n(Y, y)$ is an isomorphism.*
(3) *If (X, x) is a monoid up to homotopy in* Top_**, then the group structure on $\pi_1(X, x)$ is abelian.*
(4) *A path γ from x to x' in X induces an isomorphism $\pi_n(X, x) \cong \pi_n(X, x')$ for all $n \geq 1$.*

In particular, for $n \geq 2$, $\pi_n(X, x)$ is abelian.

Proof. (1) The evaluation at the non-base point -1 of S^0 gives a homeomorphism $\mathrm{Map}_*(S^0, X) \to X$, so the claim follows. (2) The canonical map $\mathrm{Map}_*(S^n, X \times Y) \to \mathrm{Map}_*(S^n, X) \times \mathrm{Map}_*(S^n, Y)$ is a homeomorphism since S^n is locally compact, see Remark 1.108. The claim then follows from the fact that π_0 commutes with finite products. (3) Since the forgetful functor $\mathrm{Grp} \to \mathrm{Set}$ commutes with finite products and is conservative, we see that $\pi_1 \colon \mathrm{Top}_* \to \mathrm{Grp}$ commutes with finite products. It follows that the image of a monoid up to homotopy in Top_* is sent to a monoid in Grp. The Eckmann–Hilton argument below shows that this makes the group structure on π_1 commutative. The "in particular" follows from the above observed isomorphism $\pi_n(X, x) \cong \pi_1(\Omega_x(\mathrm{Map}_*(S^{n-2}, X)))$. (4) $\pi_n(X, x) \cong \pi_{n-1}(\Omega_x(X))$, so the claim follows from the fact that π_{n-1} is a pointed homotopy invariant functor together with the pointed homotopy equivalence $\Omega_x(X) \simeq \Omega_{x'}(X)$ from Lemma 2.20. □

2.25. **Addendum** The Eckmann-Hilton argument is a slight generalisation of what we have used above. It says the following. Suppose M is a set equipped with unital binary operations $\circ$ and $\star$ with units $e_\circ$ and $e_\star$ and which satisfy that the diagram

$$\begin{array}{ccc} M \times M \times M \times M & \xrightarrow{(\circ\times\circ)(\mathrm{id}\times\tau\times\mathrm{id})} & M \times M \\ \downarrow{\scriptstyle \star\times\star} & & \downarrow{\scriptstyle \star} \\ M \times M & \xrightarrow{\circ} & M \end{array}$$

commutes. Then $\circ = \star$ and both operations are associative and commutative. Indeed, the commutativity of the diagram says that for all $x, y, z, w \in M$ we have $(x \star y) \circ (z \star w) = (x \circ z) \star (y \circ w)$. Then we get

$$e_\star = (e_\circ \circ e_\star) \star (e_\star \circ e_\circ) = (e_\circ \star e_\star) \circ (e_\star \star e_\circ) = e_\circ$$

so $e_\star = e_\circ =: e$. In addition,

$$x \circ w = (x \star e) \circ (e \star w) = (x \circ e) \star (e \circ w) = x \star w$$

so $\circ = \star$. Moreover,

$$x \circ w = (x \star e) \circ (e \star w) = (e \star x) \circ (w \star e) = (e \star w) \circ (x \star e) = w \circ x$$

so $\circ$ is commutative. Finally,

$$(x \circ y) \circ w = (x \circ y) \star (w \circ e) = (x \star e) \circ (y \star w) = x \circ (y \circ w)$$

so $\circ$ is associative.

2.26. **Remark** The definition of homotopy groups we have given can also be done by replacing $(S^n, 1)$ with the pair $(D^n, \partial D^n)$ and can also be phrased as follows: $\pi_n(X, x)$ is the set of based homotopy classes of based maps $(S^n, 1) \to (X, x)$ or equivalently of homotopy classes rel ∂D^n of maps $D^n \to X$ sending ∂D^n to x. Indeed, this follows from Remark 2.3 since S^n and D^n are locally compact. The latter formulation is a frequently used definition of homotopy groups.

2.27. **Definition** Let $f\colon X \to Y$ be a continuous map between topological spaces. Then f is called a *weak homotopy equivalence* if for each $x \in X$ and $n \geq 0$ it induces a bijection $\pi_n(X, x) \to \pi_n(Y, f(x))$.

2.28. **Example** A pointed homotopy equivalence is a weak homotopy equivalence. Indeed, this follows simply from the fact that π_n is a pointed homotopy invariant functor.

2.29. **Lemma** *A homotopy equivalence $f\colon X \to Y$ is a weak homotopy equivalence.*

Proof. The map $\pi_0(f)$ is a bijection, so it remains to show that for all $x \in X$, the map $\pi_n(f)\colon \pi_n(X, x) \to \pi_n(Y, f(x))$ is a bijection. To do so, we first consider the following general fact. Let $H\colon X \times [0, 1] \to Y$ be a homotopy from f to f'. The path $\{x\} \times [0, 1] \to X \times [0, 1]$ induces a commutative diagram

$$\begin{array}{ccccc} \pi_n(X, x) & \overset{i_0}{\underset{\longleftarrow}{\longrightarrow}} & \pi_n(X \times [0,1], (x, 0)) & \xrightarrow{H} & \pi_n(Y, f(x)) \\ \| & & \downarrow \cong & & \downarrow \cong \\ \pi_n(X, x) & \overset{i_1}{\underset{\longleftarrow}{\longrightarrow}} & \pi_n(X \times [0,1], (x, 1)) & \xrightarrow{H} & \pi_n(Y, f'(x)) \end{array}$$

whose vertical arrows are the isomorphisms obtained in an earlier exercise, induced from the path $\{x\} \times [0, 1]$ in $X \times [0, 1]$ and its image under H in Y. The left square (with horizontally left pointing maps) is the same square as the right square with the map H replaced by the projection. Since the projection induces isomorphisms on π_n and i_0 and i_1 are sections of the projection, the left square with right pointing maps is also commutative. Finally, observe that the horizontal composites are given by $\pi_n(f)$ and $\pi_n(f')$. We deduce that if f is a homotopy equivalence and g a homotopy inverse, then the triangle

$$\begin{array}{ccc} \pi_n(X,x) & \xrightarrow{\mathrm{id}_*} & \pi_n(X,x) \\ & \searrow^{(gf)_*} & \downarrow \cong \\ & & \pi_n(X,gf(x)) \end{array}$$

commutes. With the same argument, we deduce that in the composite

$$\pi_n(X,x) \xrightarrow{f_*} \pi_n(Y,(f(x)) \xrightarrow{g_*} \pi_n(X,gf(x)) \xrightarrow{f_*} \pi_n(Y,fgf(x))$$

any two composable maps are bijections. We deduce that the map in the middle is bijective, and hence also the first map is bijective. $\square$

2.30. **Example** There are weak homotopy equivalences which are not homotopy equivalences. For instance, consider the space of a converging sequence, i.e. the set $CS = \{0\} \cup \{\frac{1}{n} \mid n \geq 1\}$ equipped with the subspace topology of $\mathbb{R}$. This set is weakly homotopy equivalent to $\coprod_{n\geq 0}\{n\}$, but not homotopy equivalent to it, see Exercise Sheet 6.

So far, we know very little about homotopy groups. Let me mention the following very basic questions:

(1) What is $\pi_1(S^n)$?
(2) What is $\pi_k(S^n)$ for $k < n$?
(3) What is $\pi_n(S^n)$?
(4) What is $\pi_k(S^n)$ for $k > n$?

2.31. **Addendum** Note that the answer to (2) wants to be 0: Indeed, given a representative $f\colon S^k \to S^n$, we surely hope that f is homotopic to a map which is not surjective (this is automatic if the map f extends to a smooth map in the sense of analysis 2 from an open neighborhood of $S^k \subseteq \mathbb{R}^{k+1}$ to an open neighborhood of $S^n \subseteq \mathbb{R}^{n+1}$). Given $x \in S^n$ which is not in the image of f, we find a factorization $S^k \to S^n \setminus \{x\} \subseteq S^n$. But the stereographic projection determines an homeomorphism $S^n \setminus \{x\} \cong \mathbb{R}^n$ (which appears on Exercise Sheet 5), and $\mathbb{R}^n$ is pointed contractible. It follows that f is pointed null-homotopic.

The answer to (1) and (3) will be given in this (and possibly the following) course: It turns out that $\pi_n(S^n) \cong \mathbb{Z}$ via an explicit invariant called the degree (more precisely, there is a canonical map $\mathbb{Z} \to \pi_n(S^n)$ sending 1 to $[\mathrm{id}_{S^n}]$ which is an isomorphism with inverse given by an invariant called the degree). We will show this by induction over n. To prove the case $n = 1$, we will make use of a theorem allowing us to calculate the fundamental group of a space by means of the fundamental groups of an open cover, known as a theorem of Seifert and van Kampen – this will also prove that $\pi_1(S^n) = 0$ for $n > 1$.

Possibly surprisingly (at first glance) is that we will not be able to say much about $\pi_k(S^n)$ when $k > n$. In the beginning of the rise of algebraic topology, there were some thoughts that these groups might also be trivial, a thought that could not be further from the truth. In fact, calculating these groups is one of the major tasks of homotopy theory and our knowledge of these groups (though by now very impressive) is on a large scale still tiny, and the tools that go into calculations are very intricate and sometimes indirect.

2.2. **CW complexes.** Before moving to the theorem of Seifert and van Kampen, we briefly introduce CW complexes.

2.32. **Definition** Let X be a topological space and $A \subseteq X$ a subspace. A *CW* structure on the pair (X, A) consists of a filtration $\{\mathrm{sk}_n(X)\}_{n\geq -1}$ by subspaces, called the *skeletal filtration*, with $\mathrm{sk}_{-1}(X) = A$ such that the following holds. First, the canonical map $\mathrm{colim}_n \, \mathrm{sk}_n(X) \to X$ is a homeomorphism. Second, for each $n \geq -1$, there is a set I_n and for each $i \in I_n$ there are maps $\alpha_i\colon S^n \to \mathrm{sk}_n(X)$ such that there is a pushout diagram

$$\begin{array}{ccc} \coprod_{i\in I_n} S^n & \xrightarrow{\amalg_i \alpha_i} & \mathrm{sk}_n(X) \\ \downarrow & & \downarrow \\ \coprod_{i\in I_n} D^{n+1} & \xrightarrow{\amalg_i \beta_i} & \mathrm{sk}_{n+1}(X) \end{array}$$

Here, $S^{-1} = \varnothing$. A CW complex (X, A) is short hand for a CW structure on the pair (X, A); often A is taken to be empty in which case we simply refer to X as a CW complex.

It is important to remember that a CW structure is a space equipped with a filtration, such that some property holds (colimit topology and existence of pushouts). The pushouts are not part of the data, only the existence is required. A space can have no or many CW structures. We will often refer to a CW complex X, and then mean that X is a topological space equipped with a CW filtration $\{\mathrm{sk}_n(X)_{n\geq 0}\}$ which we leave implicit.

2.33. **Example**

(1) The spheres S^n are CW complexes via the filtration

$$\mathrm{sk}_k(S^n) = \begin{cases} \{x\} & \text{if } -1 < k < n \\ S^n & \text{if } k \geq n \end{cases}.$$

Here, x is an arbitrary basepoint. Indeed, this follows from the homeomorphism $D^n/S^{n-1} \cong S^n$.

(2) The disk D^n is a CW complex with filtration $\emptyset \subseteq \{x\} \subseteq S^{n-1} \subseteq D^n$. Here, as above $S^{n-1} = \mathrm{sk}_{n-1}(D^n)$.

It turns out that many more spaces are CW complexes, we shall construct more examples later and on the exercise sheets.

We introduce the following terminology.

2.34. **Definition** Let X be a CW complex with filtration $\{\mathrm{sk}_n(X)\}_{n\geq -1}$. Then X is called

(1) N-dimensional if $\mathrm{sk}_{N'}(X) = \mathrm{sk}_N(X)$ for all $N' \geq N$.
(2) finite-dimensional if it is N-dimensional for some $N \geq -1$.
(3) of finite type, if for all $n \geq -1$, $\mathrm{sk}_{n+1}(X)$ is obtainable from $\mathrm{sk}_n(X)$ by attaching finitely many cells, i.e. there exists a pushout as in Definition 2.32 with I_{n+1} finite.
(4) finite, if it is finite dimensional and of finite type, i.e. there exists pushouts as in Definition 2.32 such that the total number of cells is finite.
(5) locally finite, if there exists pushouts as in Definition 2.32 such that each cell of X (i.e. a subset of the form $\beta_i(D^{n+1}) \subseteq X$) is disjoint from all but finitely many cells of X.

The following proposition lists some of the point-set theoretic properties of CW complexes. We will not prove it in this course.

2.35. **Proposition** *Let X be a CW complex. Then*

(1) *Every compact subset of X lies in a finite sub complex of X.*
(2) *X is compact if and only if X is finite.*
(3) *X is locally compact if and only if X is locally finite.*
(4) *X is paracompact (we have not introduced this notion yet, but it means that any open cover admits a locally finite subcover. A cover is called locally finite if for every point, there is an open neighborhood of it whose intersection with the members of the covering is empty in all but finitely many cases).*
(5) *X is normal and Hausdorff. (In fact, for paracompact spaces, Hausdorff and normal are equivalent conditions).*
(6) *X is locally path connected (in fact, it is locally contractible, that is, for every point and every open, there exists a smaller open which is contractible).*
(7) *if Y is another CW complex which is locally compact, then $X \times Y$ admits a CW structure with*

$$\mathrm{sk}_n(X \times Y) = \bigcup_{k+l=n} \mathrm{sk}_k(X) \times \mathrm{sk}_l(Y).$$

We move on to maps between CW complexes.

2.36. **Definition** Let X and Y be CW complexes. A map $f\colon X \to Y$ is called *cellular* if $f(\mathrm{sk}_n(X)) \subseteq \mathrm{sk}_n(Y)$ for all $n \geq -1$.

Again, whether or not a map is cellular very much depends on the CW filtration one has chosen. In particular, (somewhat obviously) not every map is cellular. The following result, called *cellular approximation*, is technically important and will be proven most likely in the next course.

2.37. **Theorem** (Cellular Approximation) *Let X and Y be CW complexes. Given a map $f\colon X \to Y$ satisfying $f(\mathrm{sk}_k(X)) \subseteq \mathrm{sk}_k(Y)$ for some $k \geq -1$, there exists a homotopy rel $\mathrm{sk}_k(X)$ from f to a map f' satisfying $f'(\mathrm{sk}_n(X)) \subseteq \mathrm{sk}_n(Y)$ for all $n \geq k$. In particular, every map between CW complexes is homotopic to a cellular map.*

Let us draw a number of useful consequences of this theorem.

2.38. **Corollary** *For $k < n$ we have $\pi_k(S^n) = 0$.*

Proof. Represent an element by a pointed map $(S^k, x) \to (S^n, y)$. We Give S^k and S^n the CW structures from Example 2.33. By cellular approximation, f is homotopic rel $\{x\}$ to a cellular map. But $\mathrm{sk}_k(S^k) = S^k$ and $\mathrm{sk}_k(S^n) = \{y\}$, showing that f is pointed homotopic to the constant map. □

More generally we have the following:

2.39. **Corollary** *Let X be a CW complex. Then for all $x \in \mathrm{sk}_0(X)$, the induced map $\pi_k(\mathrm{sk}_n(X), x) \to \pi_k(X, x)$ is an isomorphism for $k \leq n-1$ and a surjection for $k = n$.*

Proof. Surjectivity for $k \leq n$: Consider a pointed map $f\colon (S^k, 1) \to (X, x)$ representing an element of $\pi_k(X, x)$. By cellular approximation there is a homotopy rel $\{x\}$, i.e. a pointed homotopy, from f to a cellular map f'. In particular $f'\colon (S^k, 1) \to (\mathrm{sk}_k(X), x)$, so surjectivity follows. To see injectivity when $k < n$, assume given pointed maps $f, g\colon S^k \to \mathrm{sk}_k(X)$ and a pointed homotopy $H\colon S^k \to X$ from f to g. Recall from Proposition 2.35 (7) that $S^k \times [0,1]$ has a CW structure with $\mathrm{sk}_k(S^k \times [0,1]) = S^k \times \{0,1\} \cup \{1\} \times [0,1]$ and that $H_{|\{1\}\times[0,1]}$ is constant at x. In particular, with respect to the just mentioned CW structure on $S^k \times [0,1]$, the map H sends $\mathrm{sk}_k(S^k \times [0,1])$ to $\mathrm{sk}_k(X)$. By cellular approximation, H is homotopic rel $S^k \times \{0,1\} \cup \{1\} \times [0,1]$ to a map with image contained in $\mathrm{sk}_{k+1}(X)$. Since $k+1 \leq n$, injectivity of the map under investigation follows. □

2.40. **Corollary** *For every space X, there exists a weak homotopy equivalence $A \to X$ where A is a CW complex.*

Proof. We argue for each path component of X that there is a connected CW complex with a weak homotopy equivalence to that path component. Then we obtain a map from the coproduct of all these CW complexes to X which is a weak homotopy equivalence. The result follows then from the observation that coproducts of CW complexes are CW complexes.

So let us assume that X is path connected and choose a basepoint $x \in X$. Choose generators $\{\alpha_i\}_{i\in I}$ of $\pi_1(X, x)$. They induce a pointed map

$$f_1\colon \bigvee_{i\in I} S^1 \xrightarrow{\vee\alpha_i} X$$

whose composite with the canonical inclusion $\iota_j\colon S^1 \to \bigvee_{i\in I} S^1$ is given by α_i. We deduce that f_1 induces a bijection on π_0 and a surjection on π_1. We now assume inductively that we have constructed a map $f_n\colon \mathrm{sk}_n(A) \to X$ from an n-dimensional CW complex $\mathrm{sk}_n(A)$ which induces an isomorphism on π_k for $k < n$ and a surjection for $k = n$. For the induction start, we take $\mathrm{sk}_1(A) = \bigvee_{i\in I} S^1$ with its map f_1 to X. Pick a set of generators of the kernel of the map $\pi_n(A, a) \to \pi_n(X, x)$ and represent these generators by maps $\beta_j\colon S^n \to A$. Since $\beta_j \in \ker(\pi_n(A, a) \to \pi_n(X, x))$, there exists a null homotopy H of $f_n\beta_j$. Such a null homotopy H is in particular a map $S^n \times [0,1]$ whose restriction to $S^n \times \{1\}$ is constant. H therefore induces a map $\bar{H}_j\colon D^{n+1} \to X$ whose restriction to the boundary S^n is given by $f_n\beta_j$, since $S^n \times [0,1]/S^n \times \{1\} \cong D^{n+1}$. Consequently, there exists a commutative diagram

$$\begin{array}{ccc} \coprod_{j\in J} S^n & \xrightarrow{\amalg\beta_j} & \mathrm{sk}_n(A) \\ \downarrow & & \downarrow{\scriptstyle f_n} \\ \coprod_{j\in J} D^{n+1} & \xrightarrow{\amalg\bar{H}_j} & X \end{array}$$

Define $\mathrm{sk}_{n+1}(A)'$ to be the pushout of the upper-left part of the above diagram. Then $\mathrm{sk}_{n+1}(A)'$ is an $(n+1)$-dimensional CW complex with $\mathrm{sk}_k(\mathrm{sk}_{n+1}(A)') = \mathrm{sk}_k(A)$ for $k \leq n$ and we obtain a unique map $f'_{n+1} \to X$ extending the map f_n by the universal property of the pushout. We claim that f'_{n+1} induces an isomorphism on π_k for $k \leq n$. To see this, we consider the composite

$$\pi_k(\mathrm{sk}_n(A), a) \xrightarrow{i_n} \pi_k(\mathrm{sk}_{n+1}(A)', a) \xrightarrow{f'_{n+1}} \pi_k(X, x)$$

which is given induced by f_n. Hence, for $k < n$, the composite is an isomorphism by the inductive hypothesis and the first map is an isomorphism by Corollary 2.39. For $k = n$, the composite is surjective by the inductive hypothesis, so the latter map is also surjective.

Again by Corollary 2.39, the first map is also surjective. Moreover, the kernel of the composite is contained in the kernel of the first map (by construction). This implies that the second map is also injective and hence an isomorphism. Now choose a set of generators $\{\gamma_k\}_{k\in K}$ of $\pi_{n+1}(X,x)$ and define the space

$$\mathrm{sk}_{n+1}(A) = \mathrm{sk}_{n+1}(A)' \vee \bigvee_{k\in K} S^{n+1}.$$

Then $\mathrm{sk}_{n+1}(A)$ is again an $(n+1)$-dimensional CW complex with $\mathrm{sk}_k(\mathrm{sk}_{n+1}(A)) = \mathrm{sk}_k(A)$ for $k \leq n$. The map f'_{n+1} together with the maps γ_k induce a unique map $f_{n+1}\colon \mathrm{sk}_{n+1}(A) \to X$. This map is surjective on π_{n+1} (this is clear by construction) and an isomorphism on π_k for $k \leq n$. Indeed, the composite

$$\pi_k(\mathrm{sk}_{n+1}(A)', a) \to \pi_k(\mathrm{sk}_{n+1}(A), a) \xrightarrow{f_n} \pi_k(X, x)$$

is an isomorphism by what we have just argueed, showing that the second map is surjective. The first map is also surjective by Corollary 2.39, and injective because it has a retraction induced by the unique map $\bigvee S^{n+1} \to *$. Hence, the first map is an isomorphism, as well as the composite, and hence so is the latter map. This finishes the inductive step. Then we let $A = \mathrm{colim}_n \,\mathrm{sk}_n(A)$ and obtain a unique continuous map $f\colon A \to X$ restricting to the previously constructed f_n on $\mathrm{sk}_n(A)$. Then, for any $k \geq 0$, we choose $n > k$ and then have that of the following maps

$$\pi_k(\mathrm{sk}_n(A), a) \to \pi_k(A, a) \to \pi_k(X, x)$$

the composite and the first map are isomorphisms. Hence the latter map is also an isomorphism, showing that f is a weak homotopy equivalence. □

2.41. **Remark** In this course, and in fact also the following courses, we will almost exclusively investigate invariants of topological spaces which are not only homotopy invariant but in fact invariant under weak homotopy equivalences, like homotopy groups and also the singular (co)homology groups which we will introduce in due time. The previous corollary hence implies that we may replace any topological space by a weakly equivalent CW complex without changing its invariants, or in other words, through the eyes of this invariant, we may restrict our attention to CW complexes to begin with.

The typical invariant that distinguishes weakly equivalent spaces which are not homotopy equivalent (hence spaces necessarily not homotopy equivalent to CW complexes) is sheaf cohomology (rather than singular cohomology). While itself a very interesting invariant with many applications, we will not discuss it in this course. Perhaps one can have a seminar about sheaf cohomology at some point, and its relation to singular cohomology for suitable spaces.

The following theorem says that the relation of weak homotopy equivalences restricted to CW complexes is indeed the relation of homotopy equivalences. We will prove it next term when we introduce a further important notion in homotopy theory, namely that of a fibration.

2.42. **Theorem** (Whitehead) *Let $f\colon X \to Y$ be a weak homotopy equivalence and A a CW complex. Then f induces a bijection $[A, X] \to [A, Y]$ on homotopy classes of maps out of A. In particular, if X and Y are CW complexes, then f is a homotopy equivalence.*

The in particular follows readily from Yoneda's lemma in the full subcategory of hTop on objects represented by CW complexes.

We will introduce one aspect which is relevant for the proof of Whitehead's theorem, which is that CW pairs have the *homotopy extension property.* The notion of homotopy extension properties makes sense in many other contexts as well (as soon as there is some notion of a homotopy) and we will see such a property in action in the category of groupoids when discussing the theorem of Seifert van Kampen.

2.43. **Definition** Let $i\colon A \to X$ be a map of topological spaces. We say that i is a *cofibration* if it has the homotopy extension property, i.e. given a map $f\colon X \to Y$ and $H\colon A \times [0,1] \to Y$ such that $f_{|A} = H(-,0)$, there exists a map $\bar{H}\colon X \times [0,1] \to Y$ extending H.

2.44. **Remark** The data of f and H are given by a map $A \times [0,1] \amalg_{A\times\{0\}} X \times \{0\} \to Y$. Being a cofibration then says that there exists a dashed arrow rendering the following diagram commutative:

$$\begin{array}{ccc} A \times [0,1] \amalg_{A\times\{0\}} X \times \{0\} & \xrightarrow{f \amalg H} & Y \\ \downarrow & \overset{\bar{H}}{\nearrow} & \\ X \times [0,1] & & \end{array}$$

Considering $Y = A \times [0,1] \amalg_{A\times\{0\}} X \times \{0\}$, we find that $i\colon A \to X$ is a cofibration if and only if the vertical map in the above diagram admits a retraction.

Yet another equivalent way of formulating this is that there exists a dashed arrow rendering the following diagram commutative (this uses that $[0,1]$ is locally compact):

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Map}([0,1],Y) \\ \downarrow & \overset{\bar{H}}{\nearrow} & \downarrow \mathrm{ev}_0 \\ X & \xrightarrow{f} & Y \end{array}$$

This second perspective shows the following result: Given a pushout diagram of spaces

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow i' & & \downarrow i \\ X' & \longrightarrow & X \end{array}$$

such that i' is a cofibration, then i is also a cofibration. Also, it follows that if $i_j\colon A_j \to X_j$ is a family of cofibrations, then the map $(\amalg_{j\in J} Aj \to \amalg_{j\in J} X_j)$ is also a cofibration.

2.45. **Example** (1) For all $n \geq 1$, the map $S^{n-1} \to D^n$ is a cofibration.
(2) If $i\colon A \to X$ and $j\colon X \to X'$ are cofibrations, then $ji\colon A \to X'$ is a cofibration.
(3) For all spaces X and Y, the inclusion $Y \to X \amalg Y$ is a cofibration.
(4) For all spaces X, the map $X \times \{0,1\} \to X \times [0,1]$ is a cofibration.
(5) If $i\colon A \to X$ is a cofibration and Z is a locally compact space, then $A \times Z \to X \times Z$ is a cofibration.

The proof is an exercise on Sheet 6. I will insert a proof here myself after the sheet has been discussed in the exercise sessions.

2.46. **Theorem** (Homotopy extension property) *Let (X,A) be a CW pair. Then the inclusion $A \subseteq X$ is a cofibration.*

Proof. The pairs (D^n, S^{n-1}) satisfy the homotopy extension property by Example 2.45. It follows from Remark 2.44 that for all $n \geq -1$, the pairs $(\mathrm{sk}_n(X), A)$ have the homotopy extension property. Then, it follows from the fact that $X = \mathrm{colim}_n \mathrm{sk}_n(X)$ that also (X, A) has the homotopy extension property. □

2.47. **Corollary** *Let $i\colon A \to X$ be a cofibration, for instance the inclusion of a CW pair (X, A). Assume that A is contractible. Then the projection map $X \to X/A$ is a homotopy equivalence.*

Proof. Since A is contractible, there exists $a \in A$ and a map $H\colon A \times [0,1] \to A$ such that $H(-,0) = \mathrm{id}_A$ and $H(-,1)$ is the map which is constant at $a \in A$. Then we have a map

$$A \times [0,1] \amalg_{A \times \{0\}} X \times \{0\} \to X$$

induced by H and the identity of X. By the homotopy extension property this map extends to a homotopy $\bar{H}\colon X \times [0,1] \to X$ from the identity to a map $\bar{H}(-,1)$. This map has the property that it sends A to $a \in A$, and hence induces a continuous map $h\colon X/A \to X$. Moreover, the composite $X \to X/A \xrightarrow{h} X$ is $\bar{H}(-,1)$ and hence homotopic to id_X via $\bar{H}$ itself. Moreover, $\bar{H}$ induces a map $X \times [0,1] \to X \to X/A$ which in turn induces a map $X/A \times [0,1] \to X/A$ (since $A \times [0,1] \subseteq X \times [0,1] \to X/A$ is constant at the equivalence class of A). This map is a homotopy between the identity and the composite $X/A \xrightarrow{h} X \to X/A$. □

We finish this section with some constructions that allow us to replace an arbitrary map by a cofibration, up to homotopy equivalence.

2.48. **Definition** Let $f\colon X \to Y$ be a map. We define $\mathrm{Cyl}(f)$ and maps $X \amalg Y \to \mathrm{Cyl}(f) \to Y$ via the following pushout squares

$$\begin{array}{ccccccc} X & \xrightarrow{\iota_l} & X \amalg X & \xrightarrow{i_0 \amalg i_1} & X \times [0,1] & \xrightarrow{\mathrm{pr}} & X \\ \downarrow f & & \downarrow f \amalg \mathrm{id} & & \downarrow & & \downarrow f \\ Y & \xrightarrow{\iota_l} & Y \amalg X & \xrightarrow{\iota_Y \amalg \iota_X} & \mathrm{Cyl}(f) & \xrightarrow{p} & Y \end{array}$$

2.49. **Remark** Let $i\colon A \to X$ be a map of spaces. Then $\mathrm{Cyl}(i) = A \times [0,1] \amalg_{A \times \{0\}} X \times \{0\}$. Hence, we can rewrite the condition that i is a cofibration by the condition that the tautological map $\mathrm{Cyl}(i) \to X \times [0,1]$ has a retraction.

2.50. **Lemma** *Let $f\colon X \to Y$ be a map of spaces. Then the map f factors as the composite $X \xrightarrow{\iota_X} \mathrm{Cyl}(f) \xrightarrow{p} Y$ and the maps ι_X and ι_Y are cofibrations. Moreover, the map p is a homotopy inverse of the map ι_Y. In particular, $\mathrm{Cyl}(f)$ and Y are canonically homotopy equivalent.*

Proof. The first part follows from the definitions together with Example 2.45. By construction, the composite $Y \to \mathrm{Cyl}(f) \to Y$ is the identity. It hence suffices to show that the composite $\mathrm{Cyl}(f) \to Y \to \mathrm{Cyl}(f)$ is homotopic to the identity. To do this, let $H\colon [0,1] \times [0,1] \to [0,1]$ be a map with $H(-,0) = \mathrm{id}$ and $H(-,1)$ constant at 0, e.g. $H(t,s) = (1-s)t$. Since $[0,1]$ is locally compact, the following two squares are pushout diagrams

$$\begin{array}{ccc} X\times\{0\}\times[0,1] & \longrightarrow & X\times[0,1]\times[0,1] \\ \downarrow & & \downarrow \\ Y\times[0,1] & \longrightarrow & \mathrm{Cyl}(f)\times[0,1] \end{array} \qquad \begin{array}{ccc} X\times\{0\} & \longrightarrow & X\times[0,1] \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathrm{Cyl}(f) \end{array}$$

Consider the following maps from each term of the left (upper part) of the square to the right one: $X\times\{0\}\times[0,1]\to X\times\{0\}$, the projection, $Y\times[0,1]\to Y$ the projection, and the map $\mathrm{id}_X\times H\colon X\times[0,1]\times[0,1]\to X\times[0,1]$. These maps induce, by the universal property of pushouts, a map $\bar{H}\colon \mathrm{Cyl}(f)\times[0,1]\to\mathrm{Cyl}(f)$. Then $\bar{H}(-,0)$ is the identity of $\mathrm{Cyl}(f)$, and $\bar{H}(-,1)$ is given by p. Indeed, to see this, it suffices to show that $\bar{H}(-,1)$ and p agree upon restriction along ι_Y and $X\times[0,1]\to\mathrm{Cyl}(f)$ which is true by construction. □

2.51. **Definition** Let $f\colon X\to Y$ be a map. Then we define its mapping cone $\mathrm{C}(f)$ together with maps $\mathrm{Cyl}(f)\to\mathrm{C}(f)$ and $\mathrm{C}(f)\to\Sigma(X)$ via the pushout diagrams

$$\begin{array}{ccccc} X & \xrightarrow{\iota_X} & \mathrm{Cyl}(f) & \longrightarrow & \mathrm{C}(X) \\ \downarrow & & \downarrow{\scriptstyle c} & & \downarrow \\ * & \longrightarrow & \mathrm{C}(f) & \longrightarrow & \Sigma(X) \end{array}$$

where the map $\mathrm{Cyl}(f)\to\mathrm{C}(X)$ is obtained by collapsing the image of Y inside $\mathrm{Cyl}(f)$.

2.52. **Remark** We note that the map ι_X factors as the composite $X\times\{1\}\to X\times[0,1]\to\mathrm{Cyl}(f)$. Consequently, we have the following diagram of pushout squares.

$$\begin{array}{ccccccc} & & X\times\{0\} & \xrightarrow{f} & Y & \longrightarrow & * \\ & & \downarrow & & \downarrow & & \downarrow \\ X\times\{1\} & \longrightarrow & X\times[0,1] & \longrightarrow & \mathrm{Cyl}(f) & \longrightarrow & \mathrm{C}(X) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle c} & & \downarrow \\ * & \longrightarrow & \mathrm{C}(X) & \longrightarrow & \mathrm{C}(f) & \longrightarrow & \Sigma(X) \end{array}$$

In particular the middle vertically combined square is a pushout square, showing that $\mathrm{C}(f)$ is obtained by gluing $\mathrm{C}(X)$ on Y along the map f.

2.53. **Lemma** *Let $i\colon A\to X$ be a map. Then there is a canonical map $\mathrm{C}(i)\to X/A$. This map is a homotopy equivalence if i is a cofibration.*

Proof. The previous remark says that the left of the following two squares is a pushout diagram.

$$\begin{array}{ccccc} A & \longrightarrow & \mathrm{C}(A) & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & \mathrm{C}(i) & \longrightarrow & X/A \end{array}$$

The combined square is also a pushout, hence so is the right square. Consequently, we obtain the wanted map $\mathrm{C}(i)\to X/A$. If i is a cofibration, then the left square being a pushout implies that $\mathrm{C}(A)\to\mathrm{C}(i)$ is also a cofibration. The right square being a pushout means that there is a homeomorphism $X/A \cong \mathrm{C}(i)/\mathrm{C}(A)$ such that the map $\mathrm{C}(i) \to X/A$ becomes the tautological map $\mathrm{C}(i) \to \mathrm{C}(i)/\mathrm{C}(A)$. Since $\mathrm{C}(A)$ is contractible, the lemma follows from Corollary 2.47. □

2.54. **Corollary** *Let $i\colon A \to X$ be a cofibration. Then there exists a map $X/A \to \Sigma(A)$, natural up to homotopy.*

Proof. There are natural maps $X/A \leftarrow \mathrm{C}(i) \to \Sigma(A)$ of which the first is a homotopy equivalence. Choosing a homotopy inverse (which is unique up to homotopy) of this map provides the desired map. □

2.3. **Seifert van Kampen.** We now come to the theorem of Seifert van Kampen. We will first prove a version of it where we replace the fundamental group introduced above by a slightly more flexible object, the fundamental groupoid of X. It is defined as follows.

2.55. **Definition** Let X be a topological space. Its fundamental groupoid $\tau_{\leq 1}(X)$ is the groupoid whose objects are the points of X and where

$$\mathrm{Hom}_{\tau_{\leq 1}(X)}(x, y) = \pi_0(\Omega_{x,y}(X))$$

is given by homotopy classes rel endpoint of paths $[0, 1] \to X$ from x to y. Composition is defined by concatenation of paths and appropriate reparametrisation. Units are given by constant paths. Inverses of paths are given by the same path run in opposite direction.

2.56. **Remark** For X a topological space and $x \in X$, we have an isomorphism of groups $\pi_1(X, x) \cong \mathrm{Hom}_{\tau_{\leq 1}(X)}(x, x)$. Moreover, in a groupoid $\mathcal{G}$, any morphism $f\colon x \to y$ induces an isomorphism of groups $\mathrm{Aut}_{\mathcal{G}}(x) \cong \mathrm{Aut}_{\mathcal{G}}(y)$. For a choice of path γ from x to x', viewed as element of $\mathrm{Hom}_{\tau_{\leq 1}(X)}(x, x')$, the associated isomorphism $\pi_1(X, x) \cong \mathrm{Hom}_{\tau_{\leq 1}(X)}(x, x) \cong \mathrm{Hom}_{\tau_{\leq 1}(X)}(y, y) \cong \pi_1(X, y)$ equals the one which is obtained from Lemma 2.20 by applying π_0.

The theorem we aim to prove is the following.

2.57. **Theorem** (Seifert van Kampen - Groupoid version) *Let X be a topological space and let $A, B \subseteq X$ be subspaces such that $\mathring{A} \cup \mathring{B} = X$. Then the induced diagram*

$$\begin{array}{ccc} \tau_{\leq 1}(A \cap B) & \longrightarrow & \tau_{\leq 1}(A) \\ \big\downarrow & & \big\downarrow \\ \tau_{\leq 1}(B) & \longrightarrow & \tau_{\leq 1}(X) \end{array}$$

is a pushout in Gpd.

Proof. Consider a commutative diagram of groupoids as follows:

$$\begin{array}{ccc} \tau_{\leq 1}(A \cap B) & \longrightarrow & \tau_{\leq 1}(A) \\ \big\downarrow & & \big\downarrow{\scriptstyle F_A} \\ \tau_{\leq 1}(B) & \xrightarrow{F_B} & \mathcal{G} \end{array}$$

Then we need to show that there exists a unique functor $F\colon \tau_{\leq 1}(X) \to \mathcal{G}$ whose restriction to $\tau_{\leq 1}(A)$ is F_A and whose restriction to $\tau_{\leq 1}(B)$ is F_B. To define F we need to evaluate it on objects and morphisms. Evaluating on objects is easy: pick $x \in X$. If $x \in A$, then define $F(x) = F_A(x)$. If $x \in B$, define $F(x) = F_B(x)$. If $x \in A \cap B$, then $F_A(x) = F_B(x)$ by the commutativity of the above diagram, so this association is well-defined. So let us try to define F on morphisms, i.e. on a path $\gamma\colon [0,1] \to X$ from x to x'. Consider the open cover $[0,1] = \gamma^{-1}(A) \cap \gamma^{-1}(B)$. By definition of the topology on $[0,1]$, for each $t \in [0,1]$ there exists an $\epsilon_t > 0$ such that the closed ϵ_t ball (aka interval) around t is contained in $\gamma^{-1}(A)$ or $\gamma^{-1}(B)$. Since $[0,1]$ is compact, we can find finitely many such intervals which cover $[0,1]$. As a consequence, we can find $0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1$ such that $\gamma([t_i, t_{i+1}])$ is contained in A or B. In particular, $\gamma = \gamma_n \star \cdots \star \gamma_1$, where γ_i is γ restricted to $[t_{i-1}, t_i]$ and reparametrized appropriately. Since F is supposed to be a functor we must define $F(\gamma) = F(\gamma_n) \circ \cdots \circ F(\gamma_1)$. Since γ_i is a path in either A or B, and F restricted to $\tau_{\leq 1}(A)$ is supposed to be equal to F_A, and restricted to $\tau_{\leq 1}(B)$ is supposed to be F_B, $F(\gamma)$ is determined by these properties. It remains to show that this definition is well-defined and indeed gives a functor $F\colon \tau_{\leq 1}(X) \to \mathcal{G}$. For well-definedness, we need to show that $F(\gamma) = F(\gamma')$ if γ is homotopic rel endpoints to γ'. Pick a homotopy rel endpoints $H\colon [0,1] \times [0,1] \to X$ from γ to γ'. By a similar argument as above, we can find an $N \geq 0$ such that $[0,1] \times [0,1]$ is the union of N^2 many small squares of side length $\frac{1}{N}$ (in the obvious fashion) and such that H sends each such small square to A or to B. Since F_A and F_B are well-defined, we see that so is F. Finally, we need to see that the so defined F is a functor. Let $x \in A$. Then $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$ since id_x is the constant path at x, hence contained in A so that $F(\mathrm{id}_x) = F_A(\mathrm{id}_x) = \mathrm{id}_{F_A}(x) = \mathrm{id}_F(x)$. Similarly one argues when $x \in B$. Now suppose γ and γ' are composable paths. Write γ and γ' as compositions of paths which lie in A or B. Then $\gamma' \star \gamma$ is the composition of these many small paths all of which lie in A or in B. Hence, $F(\gamma' \star \gamma)$ is given by applying F to all small pieces and composing. The same is true for $F(\gamma') \circ F(\gamma)$, showing that F is indeed a functor.

□

We will show that $\tau_{\leq 1}(X)$ has a further universal property which we intend to exploit in the following. To formulate it, we need one bit of notation.

2.58. **Construction** Given functors $\mathcal{C}_1 \xrightarrow{F_1} \mathcal{C}_0 \xleftarrow{F_2} \mathcal{C}_2$, let us define a category $\mathcal{C}_1 \hat{\times}_{\mathcal{C}_0} \mathcal{C}_2$ as follows. The objects are triples (X_1, Y_1, α) where $X_i \in \mathcal{C}_i$ and $\alpha\colon F_1(X_1) \to F_2(X_2)$ is an isomorphism. A morphism from (X_1, X_2, α) to (Y_1, Y_2, β) consists of a morphism $f_i\colon X_i \to Y_i$ such that the diagram

$$\begin{array}{ccc} F_1(X_1) & \xrightarrow{\alpha} & F_2(X_2) \\ \downarrow{\scriptstyle F_1(f_1)} & & \downarrow{\scriptstyle F_2(f_2)} \\ F_1(Y_1) & \xrightarrow{\beta} & F_2(Y_2) \end{array}$$

commutes.

Note that the full subcategory of $\mathcal{C}_1 \hat{\times}_{\mathcal{C}_0} \mathcal{C}_2$ on triples $(X_1, X_2, \mathrm{id}_{F_1(X_1)})$ canonically identifies with the pullback in the category of categories Cat of the diagram $\mathcal{C}_1 \to \mathcal{C}_0 \leftarrow \mathcal{C}_2$.

The construction $\mathcal{C}_1 \hat{\times}_{\mathcal{C}_0} \mathcal{C}_2$ is often called the 2-categorical pullback of the diagram $\mathcal{C}_1 \to \mathcal{C}_0 \leftarrow \mathcal{C}_2$.

2.59. **Lemma** *Suppose given a commutative diagram of functors*

$$\begin{array}{ccccc} \mathcal{C}_1 & \xrightarrow{F_1} & \mathcal{C}_0 & \xleftarrow{F_2} & \mathcal{C}_2 \\ \downarrow{\scriptstyle L_1} & & \downarrow{\scriptstyle L_0} & & \downarrow{\scriptstyle L_2} \\ \mathcal{D}_1 & \xrightarrow{G_1} & \mathcal{D}_0 & \xleftarrow{G_2} & \mathcal{D}_2 \end{array}$$

It induces a canonical functor $\mathcal{C}_1 \hat{\times}_{\mathcal{C}_0} \mathcal{C}_2 \to \mathcal{D}_1 \hat{\times}_{\mathcal{D}_0} \mathcal{D}_2$. *This functor is*

(1) *fully faithful if* L_i *is fully faithful for all* $i = 0, 1, 2,,$
(2) *essentially surjective if* L_1 *and* L_2 *are essentially surjective and* L_0 *is fully faithful.*

In particular, it is an equivalence of categories if L_i *is an equivalence of categories for all* $i = 0, 1, 2$.

Proof. Exercise. □

2.60. **Definition** A functor $F\colon \mathcal{C} \to \mathcal{C}'$ is called an *isofibration* if for all objects $X \in \mathcal{C}$ and isomorphisms $\alpha\colon F(X) \cong X'$ in $\mathcal{C}'$ there exists an isomorphism $\bar{\alpha}\colon X \to \bar{X}$ such that $F(\bar{\alpha}) = \alpha$.

2.61. **Example** Let $\mathcal{G}_0 \to \mathcal{G}_1$ be a functor between small groupoids which is injective on objects and let $\mathcal{G}$ be another groupoid. Then the restriction functor $\mathrm{Fun}(\mathcal{G}_1, \mathcal{G}) \to \mathrm{Fun}(\mathcal{G}_0, \mathcal{G})$ is an isofibration. Indeed, unwinding the definitions, we need to show that any functor $\mathcal{G}_0 \times \Delta^1 \amalg_{\mathcal{G}_0 \times \{0\}} \mathcal{G}_1 \times \{0\} \to \mathcal{G}$ can be extended to a functor $\mathcal{G}_1 \times \Delta^1 \to \mathcal{G}$, i.e. that $\mathcal{G}_0 \to \mathcal{G}_1$ satisfies a version of a homotopy extension property. This is left as an exercise.

2.62. **Lemma** *Let* $\mathcal{C}_1 \xrightarrow{F_1} \mathcal{C}_0 \xleftarrow{F_2} \mathcal{C}_2$ *be functors and assume that* F_1 *is an isofibration. Then the canonical functor* $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2 \to \mathcal{C}_1 \hat{\times}_{\mathcal{C}_0} \mathcal{C}_2$ *is an equivalence of categories.*

Proof. We have already observed that the canonical functor is fully faithful. To see essential surjectivity, consider an object $(X_1, X_2, \alpha\colon F_1(X_1) \cong F_2(X_2))$. Since F_1 is an isofibration, we find an isomorphism $\bar{\alpha}\colon X_1 \to X_1'$ with $F(\bar{\alpha}) = \alpha$, in particular we find $F_1(X_1') = F_2(X_2)$. Consider then the object $(X_1', X_2, \mathrm{id}_{F_1(X_1)})$. Moreover, pair $(\bar{\alpha}, \mathrm{id}_{X_2})$ defines an isomorphism from (X_1, X_2, α) to (X_1', X_2, id). □

2.63. **Corollary** *For any groupoid* $\mathcal{G}$, *the diagram appearing in Theorem 2.57 induces a natural equivalence of groupoids*

$$\mathrm{Fun}(\tau_{\leq 1}(X), \mathcal{G}) \longrightarrow \mathrm{Fun}(\tau_{\leq 1}(B), \mathcal{G}) \hat{\times}_{\mathrm{Fun}(\tau_{\leq 1}(A \cap B), \mathcal{G})} \mathrm{Fun}(\tau_{\leq 1}(A), \mathcal{G}).$$

Proof. First, we claim that the pushout diagram of Theorem 2.57 induces an isomorphism of functor categories

$$\mathrm{Fun}(\tau_{\leq 1}(X), \mathcal{G}) \cong \mathrm{Fun}(\tau_{\leq 1}(B), \mathcal{G}) \times_{\mathrm{Fun}(\tau_{\leq 1}(A \cap B), \mathcal{G})} \mathrm{Fun}(\tau_{\leq 1}(A), \mathcal{G}).$$

On the level of objects, this follows immediately from Theorem 2.57. For morphisms, we use that the functor $- \times \Delta^1$ preserves pushouts (it has a right adjoint given by $\mathrm{Fun}(\Delta^1, -)$). Hence the square of Theorem 2.57 remains a pushout upon applying $- \times \Delta^1$. This shows that natural transformations between functors $\tau_{\leq 1}(X) \to \mathcal{G}$ are also as claimed. Finally, we use Lemma 2.62 together with Example 2.61, observing that $\tau_{\leq 1}(A \cap B) \to \tau_{\leq 1}(B)$ is injective on objects. □

2.64. **Corollary** (Seifert van Kampen - Group version) *Let X be a topological space and let $A, B \subseteq X$ be subspaces such that $\mathring{A} \cup \mathring{B} = X$. Assume that $A \cap B$, A, B are path connected and choose a basepoint $x \in A \cap B$. Then the induced diagram*

$$\begin{array}{ccc} \pi_1(A \cap B, x) & \longrightarrow & \pi_1(A, x) \\ \big\downarrow & & \big\downarrow \\ \pi_1(B, x) & \longrightarrow & \pi_1(X, x) \end{array}$$

is a pushout in Grp. *Equivalently, the canonical map $\pi_1(B, x) *_{\pi_1(A \cap B, x)} \pi_1(A, x) \to \pi_1(X, x)$ is an isomorphism of groups.*

Proof. It suffices to show that the composite

$$B(\pi_1(B, x) \star_{B\pi_1(A \cap B, x)} \pi_1(A, x)) \longrightarrow B\pi_1(X, x) \to \tau_{\leq 1}(X)$$

is an equivalence of categories. To do this, we consider the commutative diagram

$$\begin{array}{ccccc} B\pi_1(B, x) & \longleftarrow & B\pi_1(A \cap B, x) & \longrightarrow & B\pi_1(A, x) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \tau_{\leq 1}(B) & \longleftarrow & \tau_{\leq 1}(A \cap B) & \longrightarrow & \tau_{\leq 1}(A) \end{array}$$

whose vertical arrows consist of equivalences by the assumptions that all spaces appearing are path-connected. For each groupoid $\mathcal{G}$, we then obtain that the canonical functor

$$\mathrm{Fun}(B\pi_1(B, x), \mathcal{G}) \hat{\times}_{\mathrm{Fun}(B\pi_1(A \cap B, x))} \mathrm{Fun}(B\pi_1(A, x), \mathcal{G}) \longrightarrow \mathrm{Fun}(\tau_{\leq 1}(B), \mathcal{G}) \hat{\times}_{\mathrm{Fun}(\tau_{\leq 1}(A \cap B), \mathcal{G})} \mathrm{Fun}(\tau_{\leq 1}(A), \mathcal{G})$$

is an equivalence of categories, see Lemma 2.59. Corollary 2.63, and its variant for the pushout diagram

$$\begin{array}{ccc} B\pi_1(A \cap B, x) & \longrightarrow & B\pi_1(A, x) \\ \big\downarrow & & \big\downarrow \\ B\pi_1(B, x) & \longrightarrow & B(\pi_1(B, x) \star_{\pi_1(A \cap B, x)} \pi_1(A, x)) \end{array}$$

hence gives a natural equivalence of categories

$$\mathrm{Fun}(B(\pi_1(B, x) \star_{B\pi_1(A \cap B, x)} B\pi_1(A, x), \mathcal{G}) \simeq \mathrm{Fun}(\tau_{\leq 1}(X), \mathcal{G}).$$

The Yoneda lemma then shows that the functor

$$B(\pi_1(B, x) \star_{B\pi_1(A \cap B, x)} \pi_1(A, x)) \longrightarrow \tau_{\leq 1}(X)$$

is indeed an equivalence of categories as needed. ☐

2.65. **Corollary** *There is an isomorphism $\pi_1(S^1) \cong \mathbb{Z}$.*

Proof. First, we calculate $\tau_{\leq 1}(S^1)$ as follows. We cover S^1 by two intervals with intersection given by the disjoint union of two intervals. Using the above results, we then obtain a natural equivalence

$$\mathrm{Fun}(\tau_{\leq 1}, \mathcal{G}) \simeq \mathrm{Fun}(*, \mathcal{G}) \hat{\times}_{\mathrm{Fun}(* \amalg *, \mathcal{G})} \mathrm{Fun}(*, \mathcal{G}) \simeq \mathcal{G} \hat{\times}_{\mathcal{G} \times \mathcal{G}} \mathcal{G}$$

since $\tau_{\leq 1}((0,1)) \simeq *$, and $\tau_{\leq 1}((0,1) \amalg (0,1)) \simeq * \amalg *$. In the last formula, the functors that appear are the diagonal functors. Now, let us calculate this final term: It is a groupoid whose objects are given by triples $(x, y, (\alpha, \beta))$ where $\alpha, \beta \colon x \to y$ are isomorphisms, and where a morphism to $(x', y', (\alpha', \beta'))$ are given by tuples (f, g) where $f \colon x \to x'$ and $g \colon y \to y'$ are isomorphisms intertwining α, α' and β, β'. Now consider the category $\mathrm{Fun}(B\mathbb{Z},\mathcal{G})$. Its objects are given by pairs (x,γ) where $\gamma\colon x\to x$ is an automorphism, and where morphisms from (x,γ) to (x',γ') are given by isomorphisms $f\colon x\to x'$ intertwining γ, γ'. We claim that there is a natural equivalence

$$\mathrm{Fun}(B\mathbb{Z},\mathcal{G})\simeq \mathcal{G}\hat{\times}_{\mathcal{G}\times\mathcal{G}}\mathcal{G}$$

given by sending (x,γ) to $(x,x,(\gamma,\mathrm{id}))$. (Exercise: the inverse is given by sending $(x,y,(\alpha,\beta)$ to $(x,\alpha^{-1}\beta))$. In total, we obtain a natural equivalence

$$\mathrm{Fun}(B\mathbb{Z},\mathcal{G})\simeq \mathcal{G}\hat{\times}_{\mathcal{G}\times\mathcal{G}}\mathcal{G}\simeq \mathrm{Fun}(\tau_{\leq 1}(S^1),\mathcal{G})$$

so that again the Yoneda lemma implies that $B\mathbb{Z}$ and $\tau_{\leq 1}(S^1)$ are equivalent. In particular, $\pi_1(S^1)\cong\mathbb{Z}$. □

2.66. **Remark** We will later see that one can make this a priori abstract isomorphism $\pi_1(S^1)\cong\mathbb{Z}$ explicit in two ways: First, we claim that the map $\mathbb{Z}\to\pi_1(S^1)$ given by sending 1 to $[\mathrm{id}_{S^1}]$ is an isomorphism. Furthermore, we will construct an invariant $\deg\colon \pi_1(S^1)\to\mathbb{Z}$ which implements an inverse to this isomorphism. The *degree*, or the *winding number* of a loop simply counts (with signs) how often the loop passes the basepoint of S^1 – one can arrange up to homotopy that $\gamma\colon S^1\to S^1$ has the property that the preimage of 1 is a discrete, and hence finite subset of S^1.

Before finishing, let us use the Seifert van Kampen theorems to calculate fundamental further spaces.

2.67. **Example** (Suspensions) Let X be a path connected topological space and recall that $\Sigma(X)$ denotes its suspension. Then $\pi_1(\Sigma(X),x)=1$. In particular, $\pi_1(S^n,1)=1$ for $n\geq 2$. Indeed, we may cover $\Sigma(X)$ by $\Sigma(X)\setminus\{N\}$ and $\Sigma(X)\setminus S$ where N and S are the north and south poles of the suspension, respectively. We claim that $\Sigma(X)\setminus\{N\}$ is homeomorphic to $C(X)$ which is contractible (Exercise). Moreover, $\Sigma(X)\setminus\{N\}\cap\Sigma(X)\setminus\{S\}$ is homeomorphic to $X\times(0,1)$ and hence homotopy equivalent to X. Since X is path connected, the group-version of Seifert van Kampen, Corollary 2.64, applies and gives a pushout diagram of groups as follows:

$$\begin{array}{ccc}\pi_1(X,x) & \longrightarrow & \pi_1(C(X),x)\\ \downarrow & & \downarrow\\ \pi_1(C(X),x) & \longrightarrow & \pi_1(\Sigma(X),x)\end{array}$$

Since $\pi_1(C(X),x)=1$, we deduce that $\pi_1(\Sigma(X),x)=1$ as well.

2.68. **Example** (Wedges) Let (X,x) and (Y,y) be pointed spaces. Assume that there exist open and contractible subsets $U\subseteq X$ and $V\subseteq Y$ containing x and y, respectively, which deformation retract onto x and y, respectively. Then the canonical map $\pi_1(X,x)\star\pi_1(Y,y)\to\pi_1(X\vee Y,[x])$ is an isomorphism. Indeed, we consider the open cover of $X\vee Y$ given by

$$\begin{array}{ccc}U\vee V & \longrightarrow & U\vee Y\\ \downarrow & & \downarrow\\ X\vee V & \longrightarrow & X\vee Y\end{array}$$

to which we want to apply Seifert van Kampen. Since U and V deformation retract to x and y, we find that $U\vee Z$ and $Z\vee V$ are pointed homotopy equivalent to Z, for any pointed space

Z. Consequently, by applying π_1 to the above pushout diagram and using Corollary 2.64, we obtain that the square

$$\begin{array}{ccc} 1 & \longrightarrow & \pi_1(Y,y) \\ \downarrow & & \downarrow \\ \pi_1(X,x) & \longrightarrow & \pi_1(X \vee Y, [x]) \end{array}$$

is a pushout.

2.69. **Remark** CW complexes satisfy the assumption of the previous example for any basepoint. This is a slight refinement of Proposition 2.35 (6). In practice, one can also verify this condition by hand. In particular, for CW complexes X and Y, the canonical map $\pi_1(X) \star \pi_1(Y) \to \pi_1(X \vee Y)$ is an isomorphism.

2.70. **Example** (Punctured complex line) Consider $\mathbb{C} \setminus \{S\}$ for a finite set S of cardinality n. This space is homotopy equivalent to $\bigvee_n S^1$ (Exercise). Note that $1 \in S^1$ indeed has a neighborhood which deformations retracts onto 1. Hence by the previous example and Corollary 2.65, we obtain $\pi_1(\mathbb{C} \setminus \{S\}) = \mathcal{F}_n$, the free group on n generators.

2.71. **Example** (Punctured complex projective line) Consider $\mathbb{CP}^1 \setminus \{S\}$ where again S is a set of cardinality n. Since $\mathbb{CP}^1 \cong S^2$, the stereographic projection implies that $\mathbb{CP}^1 \setminus \{S\} \cong \mathbb{C} \setminus \{S \setminus \{s\}\}$. We deduce that

$$\pi_1(\mathbb{CP}^1 \setminus \{S\}) \cong \mathcal{F}_{n-1}.$$

2.72. **Example** (Presentation complex) Let S be a set and let $R \subseteq \mathcal{F}_S$ be a set of words in the free group $\mathcal{F}_S$ on the elements of S. The group $G(S,R) = \mathcal{F}_S/\langle R \rangle$, where $\langle R \rangle$ denotes the smallest normal subgroup of $\mathcal{F}_S$ containing R, is called the group presented by (S,R). Concretely, $G(S,R)$ is generated by the elements of S and has relations precisely the ones generated by R. The pushout

$$\begin{array}{ccc} \amalg_{r\in R} S^1 & \xrightarrow{r} & \bigvee_{s \in S} S^1 \\ \downarrow & & \downarrow \\ \amalg_{r \in R} D^2 & \xrightarrow{\bar{r}} & C(S,R) \end{array}$$

is called the *presentation complex* for the presentation (S,R) of $G(S,R)$. Consider for each $r \in R$ the point $\bar{r}(0) \in C(S,R)$. Then we may consider the open cover of $C(S,R)$ given by

$$C(S,R) = \bigcup_{r \in R} \bar{r}(\mathring{D}^2) \cup C(S,R) \setminus \{\bigcup_{r\in R} \bar{r}(0)\}.$$

Its intersection is given by $\coprod_{r \in R} D^2 \bar{r}(\mathring{D}^2 \setminus \{0\}$. These spaces are homotopy equivalent to the three pieces defining $C(S,R)$, in a compatible way. The groupoid version of Seifert van Kampen implies then that the map $\mathcal{F}_S \cong \pi_1(\bigvee_S S^1) \to \pi_1(C(S,R))$ is surjective with kernel generated by R. Consequently, there is a canonical isomorphism $\mathcal{F}_S/\langle R \rangle \to \pi_1(C(S,R))$, hence the name presentation complex.

2.73. **Remark** The presentation complex for $\mathbb{Z}/2\mathbb{Z}$ with its obvious presentation is given by $\mathbb{RP}^2$. We will see later, that for $n \geq 2$, $\pi_n(\mathbb{RP}^2)$ is not necessarily trivial. In particular, presentation complexes can have interesting higher homotopy groups[2]. Suppose given a presentation (S, R) with $|R| = 1$, generalizing the case of $\mathbb{Z}/2\mathbb{Z}$. Such presentations are called one-relator presentations. If $\mathcal{F}_S/\langle R\rangle$ admits non-trivial torsion, e.g. if $R = T^k$ for some $k > 1$ and $T \in \mathcal{F}_S \setminus 1$, then one can show that $C(S, R)$ must again have non-trivial higher homotopy groups (we might see arguments that go into such a proof in the next terms). Conversely, it is known that if $\mathcal{F}_S/\langle R\rangle$ is torsionfree (such groups are then called torsionfree one-relator groups) the presentation complex $C(S, R)$ does *not* have non-trivial higher homotopy groups.

Furthermore, it is a conjecture (currently still open) of Whitehead that if (S, R) is a presentation such that $\pi_n(C(S, R)) = 0$ for all $n \geq 2$ and $R' \subseteq R$, then $\pi_n(C(S, R')) = 0$ for all $n \geq 2$ as well. When R consists of a single element and is such that $\mathcal{F}_S/\langle R\rangle$ is torsionfree, the conjecture predicts that $C(S, \emptyset) = \bigvee_S S^1$ has trivial higher homotopy groups. We will show that this is indeed the case in the next section on covering theory.

2.74. **Example** (Closed orientable surfaces) Let $g \geq 0$ be a natural number. Consider a regular $4g$-polygon in the plane. Identify the edges of the polygon according to the following picture to obtain the orientable genus g surface Σ_g.

On the exercise sheet, you are asked to show that there exists a pushout

$$\begin{array}{ccc} S^1 & \xrightarrow{\alpha_g} & \bigvee_{2g} S^1 \\ \big\downarrow & & \big\downarrow \\ D^2 & \longrightarrow & \Sigma_g \end{array}$$

with $\alpha = x_1y_1x_1^{-1}y_1^{-1}x_2y_2x_2^{-1}y_2^{-1}\dots x_gy_gx_g^{-1}y_g^{-1}$, thereby giving Σ_g the structure of a 2-dimensional CW complex. In other words, Σ_g is the presentation complex of the canonical presentation of the group $\mathcal{F}_{2g}/\langle x_1y_1x_1^{-1}y_1^{-1}\dots x_gy_gx_g^{-1}y_g^{-1}\rangle$, and in particular, we have

$$\pi_1(\Sigma_g) \cong \mathcal{F}_{2g}/\langle x_1y_1x_1^{-1}y_1^{-1}\dots x_gy_gx_g^{-1}y_g^{-1}\rangle.$$

Note that $\Sigma_1 \cong T^2 \cong S^1 \times S^1$. Hence $\pi_1(T^2) \cong \mathcal{F}_2/\langle xyx^{-1}y^{-1}\rangle \cong \mathbb{Z}^2 \cong \mathbb{Z} \times \mathbb{Z} \cong \pi_1(S^1) \times \pi_1(S^1)$, in line with the previously established fact that π_1 commutes with products.

3. Covering theory

3.1. **Definition** A continuous map $p\colon E \to B$ between topological spaces is called a *covering map* if there exists an open cover $\{U_i\}_{i\in I}$ of B such that for each $i \in I$ there are homeomorphisms $\theta_i\colon p^{-1}(U_i) \cong \coprod_{F_i} U_i$ where F_i is a set, such that the diagram

$$\begin{array}{ccc} p^{-1}(U_i) & \xrightarrow{\theta_i} & \coprod_{F_i} U_i \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}} & \\ & U_i & \end{array}$$

commutes, where pr denotes the canonical map which is the identity on each disjoint factor. Such an open cover is called a *trivializing open cover for p*. For each $b \in U_i$, the map θ_i induces

[2]Furthermore, it turns out that $\pi_n(C(S, R)) \neq 0$ for some $n \geq 2$ is equivalent to the condition that $\pi_2(C(S, R)) \neq 0$.

a bijection between $p^{-1}(b)$ and F_i. We will also write Fib_b^p (or Fib_b is p is understood) for $p^{-1}(b)$ and refer to it as the *fibre of* p *over* b.

3.2. **Remark** We allow that fibres are empty. In other words, the map $\varnothing \to B$ is a covering map in the above sense. As a consequence, a covering map need not be surjective. Moreover, recall that $\amalg_{F_i} U_i$ is homeomorphic to $U_i \times F_i$. Under this homeomorphism what we have called pr above is indeed the projection. Therefore, we may equivalently require that θ_i is a homeomorphism $p^{-1}(U_i) \to U_i \times F_i$.

3.3. **Example** The map $B \times F \to B$ is a covering map. It is called the *trivial covering*, whose fibres are isomorphic to F.

3.4. **Definition** Let G be a group and X a topological space. We say that a continuous action of G on X is *covering-like* if for all $x \in X$ there is an open subset $U \subseteq X$ containing x such that $gU \cap U = \varnothing$ for all $g \neq e$.

3.5. **Lemma** *Let G act continuously and freely on a space X. Then for any choice of basepoint $x \in X$, the action of G on $p^{-1}(p(x))$ is free and transitive. Moreover, the quotient map $p\colon X \to X/G$ is a covering map if and only if the action is covering-like.*

Proof. To begin, we first observe that the action of G on $p^{-1}(p(x))$ is always transitive, by definition of the quotient space X/G. Hence if the action G action on X is also free, then the action of G on $p^{-1}(p(x))$ is free and transitive as claimed.

Now assume that the action is covering-like. To see that p is a covering map, for x in X pick an open U containing x such that $gU \cap U$ is empty unless $g = e$. Since $p^{-1}(p(U)) = \bigcup_{g \in G} gU$, we see that $p(U)$ is open. The assumption that $gU \cap U$ is empty unless $g = e$ shows that $p^{-1}(p(U))$ is homeomorphic to $U \times G$, compatible with the projection to U. Hence p is a covering map.

Conversely, assume that p is a covering map. For $x \in X$ pick an open set $U \subseteq X/G$ containing $p(x)$ such that $p^{-1}(U) \cong U \times F$ for some set F, compatible with the map p. Choose $f \in F$ such that $x \in U \times \{f\}$. Now let $g \in G$ and assume that $g(U \times \{f\}) \cap U \times \{f\}$ is not empty, say x is in the intersection. Then there exists $y \in U \times \{f\}$ such that $x = gy$. In particular, we have $p(x) = p(gy) = p(y)$. Since p is injective when restricted to $U \times \{f\}$, we deduce that $x = y$, and consequently that $y = gy$. Since the G-action is assumed to be free, we find $g = e$. This shows that the action is covering-like. □

3.6. **Remark** One cannot drop the assumption that the action of G on X is free in Lemma 3.5. Indeed, a covering-like action is always free, but there are group actions such that $X \to X/G$ is a covering map without the action being free: For instance, consider the trivial action of G on X. Then X/G is homeomorphic to X, and p is a homeomorphism, in particular a covering map.

3.7. **Example**
(1) The action of $\mathbb{Z}$ on $\mathbb{R}$ by translation is covering-like. Consequently, we obtain a covering map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$. Combined with the canonical homeomorphism $\mathbb{R}/\mathbb{Z} \to S^1$, this shows that $\exp\colon \mathbb{R} \to S^1$ is a covering map.
(2) The actions of $\mathbb{Z} \rtimes \mathbb{Z}$ and $\mathbb{Z}^2$ on $\mathbb{R}$ from Exercise 3 Sheet 2 are covering-like. Consequently, we obtain covering maps $\mathbb{R}^2 \to T^2$ and $\mathbb{R}^2 \to K$. Likewise, the residual

action of C_2 on $\mathbb{R}^2/\mathbb{Z}^2 = T^2$ is covering-like, so the induced map $T^2 \to K$ is again covering map.

3.8. **Definition** We define a category $\mathrm{Cov}(B)$ whose objects are the covering maps $p\colon E \to B$ and where a morphism from $p\colon E \to B$ to $p'\colon E' \to B$ is given by a continuous map $f\colon E \to E'$ such that $p'f = p$. In other words, we define $\mathrm{Cov}(B)$ to be the full subcategory of the slice category $\mathrm{Top}_{/B}$ whose objects are covering maps. The morphisms in $\mathrm{Cov}(B)$ will be called *maps of coverings.*

The main goal of this section is to give an algebraic description of the category $\mathrm{Cov}(B)$, whenever possible. We will deviate slightly from the usual way this is done and spell out the classical results later.

3.9. **Lemma** *Let $p\colon E \to B$ be a covering map and let $f, g\colon Z \to E$ be continuous maps such that $pf = pg$. Assume that Z is connected and that there exists $z \in Z$ such that $f(z) = g(z)$. Then $f = g$.*

Proof. Consider the set $\{z \in Z \mid f(z) = g(z)\}$, i.e. the equalizer $\mathrm{Eq}(f, g)$ of f and g, which is non-empty by assumption. We have that $\mathrm{Eq}(f, g) = (f, g)^{-1}(\Delta(E))$, where $(f, g)\colon Z \to E \times_B E$ and $\Delta\colon E \to E \times_B E$ is the diagonal. Since (f, g) is continuous, $\mathrm{Eq}(f, g)$ is closed if $\Delta(E)$ is closed. We have shown in Exercise 5 Sheet 3 that this is the case since two distinct points in a fibre $p^{-1}(b)$ can indeed be separated by open subsets of E, simply by picking an neighborhood U of b such that $p^{-1}(U) \cong U \times F$. We now show that $\mathrm{Eq}(f, g)$ is also open. Indeed let $z \in \mathrm{Eq}(f, g)$. Then there exists an open neighborhood V of $f(z) = g(z)$ in E such that $p_{|V}$ is injective. Consider $V' = f^{-1}(V) \cap g^{-1}(V) \subseteq Z$, which is an open neighborhood of z. For any $z' \in V'$, to see that $f(z') = g(z')$ we may post compose with the injective map $p_{|V}$, where it is true by the assumption that $pf = pg$. Hence, $\mathrm{Eq}(f, g) \subseteq Z$ is closed, open and non-empty and hence equal to Z since Z is connected. □

The following theorem is fundamental to almost all applications and results on covering maps:

3.10. **Theorem** (Unique path lifting) *Let $p\colon E \to B$ be a covering map.*

(1) *Let $\gamma\colon [0, 1] \to B$ be a path and $e \in \mathrm{Fib}_b$. Then there exists a unique path $\bar{\gamma}_e\colon [0, 1] \to E$ such that $\bar{\gamma}(0) = e$ and $p\bar{\gamma} = \gamma$.*

(2) *If γ and γ' are homotopic rel endpoints, and $e \in \mathrm{Fib}_b$, then the lifted paths $\bar{\gamma}_e$ and $\bar{\gamma'}_e$ are homotopic rel endpoints.*

Proof. (1) Uniqueness follows from Lemma 3.9. So let us show existence. Pick a trivializing open cover $\{U_i\}$ for p. Then $\{V_i\} = \gamma^{-1}(\{U_i\})$ is an open cover of $[0, 1]$. We may therefore write $[0, 1] = [0, a_1] \cup [a_1, a_2] \cup \cdots \cup [a_{n-1}, 1]$ as a union of closed intervals $I_j = [a_{j-1}, a_j]$, such that $\gamma(I_j)$ is contained in U_i for some $i \in I$. Consequently, γ can be written as a concatenation of paths $\gamma_n \star \cdots \star \gamma_1$ such that γ_i has image in U_i. Inductively, it suffices to find a lift $\bar{\gamma}_{1e}$: Its endpoint e_2 is contained in the fibre over $\gamma_1(1)$ and serves as the new e, the starting point for a lift $\bar{\gamma}_{2e_2}$ of γ_2. Since $\{U_i\}$ is a trivializing open cover for p, we find a neighborhood V of e in E such that $p_{|V}\colon V \to U_i$ is a homeomorphism. So we define $\bar{\gamma}_{1e}$ to be $p_{|V}^{-1} \circ \gamma_1$.

(2) Let $H\colon [0, 1] \times [0, 1] \to B$ be a homotopy rel endpoints from γ to γ'. By a similar argument as above, we can find an $\epsilon > 0$ such that $[0, 1] \times [0, 1]$ is the union of small squares of side-length ϵ, and each of these small squares is sent via H to one of the open sets U_i. Again,

we wish to lift the map H inductively over all the small squares to $\bar{H}\colon [0,1]\times[0,1]\to E$. We start with the square which contains $(0,0)$. Using again that there is a neighborhood of e restricted to which p is a homeomorphism, we can lift H to $\bar{H}$ on this small square. This determines in similar fashion lifts of H to all squares which intersect the first one non-trivially. We observe that the so lifted pieces indeed combine to a continuous map since any two restrictions to the intersection of two neighboring squares are lifts of the same path in B with same start (or end) point, and hence agree by the uniqueness part of (1). In total, we obtain a lift $\bar{H}$ of H with $\bar{H}(0,0)=e$. Now, $\bar{H}(-,0)$ is a lift of the constant homotopy at b and $\bar{H}(-,1)$ is a lift of the constant homotopy at b' since H is relative endpoints (again using the uniquenss of lifts). Therefore, $\bar{H}$ is a homotopy relative endpoints from $\bar{H}(0,-)$ to $\bar{H}(1,-)$. These are lifts of $H(0,-)$ and $H(1,-)$ which are γ and γ', respectively. Moreover, $\bar{H}(0,0)=e$ by construction, and $\bar{H}(1,0)=e$ since $\bar{H}$ is relative endpoints. Again by uniqueness, $\bar{H}$ is then a homotopy rel endpoints between $\bar{\gamma}_e$ and $\bar{\gamma'}_e$. □

3.11. **Corollary** *Let $p\colon E\to B$ be a covering map and $e\in E$. Then the map $p_*\colon \pi_1(E,e)\to \pi_1(B,p(e))$ is injective. Its image consists of the loops γ at $p(e)$ whose (uniquely) lifted path $\bar{\gamma}_e$ is again a loop.*

Proof. The subset of $\pi_1(B,p(e))$ consisting of those loops which lift to closed paths is a subgroup. On this subgroup, we find an inverse by lifting. □

3.12. **Lemma** *Let G be a group which acts covering-like on a path connected space X and let $x\in X$. Then there is a short exact sequence of groups as follows.*

$$1\longrightarrow \pi_1(X,x)\xrightarrow{p_*}\pi_1(X/G,[x])\xrightarrow{\mathfrak{m}} G\longrightarrow 1.$$

In particular, if $\pi_1(X,x)=1$, then $\pi_1(X/G,[x])\cong G$.

Proof. By Corollary 3.11, we know that p_* is injective. We write $p\colon X\to X/G$ for the projection and define a map $\mathfrak{m}\colon \pi_1(X/G,[x])\to G$ as follows: Let $\gamma\colon[0,1]\to X/G$ be a closed path at $[x]$. We may lift it to a path $\bar{\gamma}\colon[0,1]\to X$ with $\bar{\gamma}(0)=x$. Then $\bar{\gamma}(1)=g(\gamma)x$ for a unique $g(\gamma)\in G$ by Lemma 3.5. Sending $[\gamma]$ to this $g(\gamma)$ is then a well-defined group homomorphism $\mathfrak{m}$. Now for $g\in G$, we may choose a path connecting x and gx since X is path connected. The image of this path is then a closed path in X/G and hence represents an element of $\pi_1(X/G,[x])$. Its image under $\mathfrak{m}$ is then given by g, showing that $\mathfrak{m}$ is surjective. Moreover, the kernel of $\mathfrak{m}$ consists of those loops $[\gamma]$ which lift to a closed loop at x in X. Again by Corollary 3.11, we find that the kernel of $\mathfrak{m}$ equals the image of p_* as needed. □

As promised, we reprove the calculation of the fundamental group of S^1:

3.13. **Example** We have $\pi_1(S^1,1)\cong\mathbb{Z}$. Indeed, we have seen earlier that the exponential function defines a homeomorphism $\mathbb{R}/\mathbb{Z}\cong S^1$, and $\mathbb{Z}$ acts covering-like on $\mathbb{R}$. Moreover, we find that a concrete isomorphism is given by lifting a loop γ at 1 in S^1 to a path in $\mathbb{R}$ starting at 0. Its endpoint is then some integer, called the *degree* of the loop γ. Hence, the degree defines an isomorphism $\pi_1(S^1,1)\to\mathbb{Z}$. Since $\deg(\mathrm{id}_{S^1})=1$, we also see that the inverse to the degree isomorphism is the map $\mathbb{Z}\to\pi_1(S^1)$ sending 1 to $[\mathrm{id}_{S^1}]$. Moreover, this map sends n to the map $x\mapsto x^n$ on S^1.

Similarly, we have $\pi_1(K,[0])\cong\mathbb{Z}\rtimes\mathbb{Z}$, as the Klein bottle was defined as the quotient $\mathbb{R}^2/\mathbb{Z}\rtimes\mathbb{Z}$, and again the action of $\mathbb{Z}\rtimes\mathbb{Z}$ on $\mathbb{R}^2$ is covering-like.

The subgroup $p_*(\pi_1(E,e))\subseteq\pi_1(B,p(e))$ is called the *characteristic subgroup of p at e*. These subgroups serve the following purpose:

3.14. **Proposition** *Let $p\colon E \to B$ be a covering map and let $f\colon X \to B$ be a continuous map with X locally and globally path connected. Then there exists a continuous map $\tilde{f}\colon X \to E$ with $p\tilde{f} = f$ if and only if for some basepoint $x \in X$, we have $f_*(\pi_1(X,x)) \subseteq p_*(\pi_1(E,e))$ for some $e \in \mathrm{Fib}^p_{f(x)}$.*

Proof. If there exists $\tilde{f}$ with $p\tilde{f} = f$, then the composite

$$\pi_1(X,x) \xrightarrow{\tilde{f}_*} \pi_1(E,\tilde{f}(x)) \xrightarrow{p_*} \pi_1(B,f(x))$$

is given by f_*, showing that $f_*(\pi_1(X,x)) \subseteq p_*(\pi_1(E,\tilde{f}(x)))$. Conversely, assume given an $e \in \mathrm{Fib}^p_{f(x)}$ such that $f_*(\pi_1(X,x)) \subseteq p_*(\pi_1(E,e))$. We define $\tilde{f}$ as follows. Let $x' \in X$ and pick a path γ from x to x'. The define $\tilde{f}(x') = \bar{f(\gamma)}_e(1)$. First, we argue that the so defined map $\tilde{f}$ is well-defined, i.e. independent of the choice of γ. So let γ' be another path from x to x'. Then $\gamma'^{-1} \star \gamma$ is a loop at x and $f(\gamma^{-1} \star \gamma)$ is therefore an element of $f_*(\pi_1(X,x))$. By assumption, this loop lifts to a closed path in E, starting (and ending) at e. This implies that $\bar{f(\gamma^{-1})}_{\tilde{f}(x)}(1) = e$, and hence that $\bar{f(\gamma)}_e = \tilde{f}(x)$. We note that, by construction, $p\tilde{f} = f$.

It remains to show that $\tilde{f}\colon X \to E$ is continuous. So let $U \subseteq E$ be open, since any such open is the union of opens of the form $\theta^{-1}(V \times \{f\})$ for some open $V \subseteq B$, we may assume that U is itself of this form, and in particular that $p_{|U}\colon U \to V$ is a homeomorphism. In this case, we find $\tilde{f}^{-1}(U) \subseteq f^{-1}(V)$. Pick $x' \in \tilde{f}^{-1}(U)$ and an open and path-connected set $V' \subseteq f^{-1}(V)$ containing x. For $y \in V'$, we may calculate $\tilde{f}(y)$ by choosing a path γ from x' to y in X, and then taking $\bar{f(\gamma)}_{\tilde{f}(x)}$. Since $f(V') \subseteq V$, a lift of $f(\gamma)$ is given by $p_{|U}^{-1}(f(\gamma))$. This shows that $V' \subseteq \tilde{f}^{-1}(U)$, and hence that $\tilde{f}$ is continuous. □

3.15. **Remark** We note that the proof above shows that if $f_*(\pi_1(X,x)) \subseteq p_*(\pi_1(E,e))$, then the constructed map $\tilde{f}$ satisfies $\tilde{f}(x) = e$. By Lemma 3.9, the map $\tilde{f}$ is uniquely determined by this property together with the requirement that $p\tilde{f} = f$.

3.16. **Corollary** *Let $p\colon E \to B$ be a covering map and $e \in E$. Then the induced map $p_*\colon \pi_n(E,e) \to \pi_n(B,p(e))$ is an isomorphism for all $n \geq 2$.*

Proof. We begin with surjectivity. So let $\alpha\colon (S^n,1) \to (B,p(e))$ be a pointed map. By Example 2.67, $\pi_1(S^n,1) = 1$ since $n \geq 2$. Therefore, Proposition 3.14 implies that α lifts to a pointed map $\bar{\alpha}\colon (S^n,1) \to (E,e)$ as needed. To see that p_* is injective, assume $\alpha,\alpha'\colon (S^n,1) \to (E,e)$ are pointed maps and that $p(\alpha)$ and $p(\alpha')$ are pointed homotopic. Choose such a pointed homotopy $H\colon S^n \times [0,1] \to B$. Since $\pi_1(S^n \times [0,1],(1,0)) = 1$, Proposition 3.14 gives a lift $\bar{H}\colon S^n \times [0,1] \to E$ with $\bar{H}(1,0) = e$. Since $H(1,-)$ is constant at $p(e)$ and $\bar{H}(1,-)$ is a lift of H with startpoint e, we deduce from the uniqueness of the lift that $\bar{H}(1,-)$ is constant at e. In particular $\bar{H}(1,1) = e$. Then we find that $\bar{H}(-,0)$ and α both lift $p(\alpha)$ and agree on $1 \in S^n$. Likewise, $\bar{H}(-,1)$ and α' both lift α' and agree on $1 \in S^n$. Therefore, $\bar{H}$ is a pointed homotopy between α and α' (by the uniqueness of lifts yet again). □

3.17. **Corollary** *For $n \geq 2$, we have $\pi_n(S^1,1) = 0$, $\pi_n(T^2,[0]) = 0$ and $\pi_n(K,[0]) = 0$. Indeed, this follows from Corollary 3.16 since $\mathbb{R}$ and $\mathbb{R}^2$ are contractible and hence have trivial homotopy groups. Likewise, we can define T^d, the d-dimensional torus as $\mathbb{R}^d/\mathbb{Z}^d$ for all $d \geq 1$. Then we find that $T^d \cong (S^1)^{\times d}$ and in particular we find $\pi_n(T^d,[0]) = 0$ for $n \geq 2$.*

3.18. **Example** The upper half plane $\mathbb{H} = \{x \in \mathbb{C} \mid \mathrm{Im}(x) > 0\}$ in the complex plane is endowed with a canonical metric, the *hyperbolic metric*. We do not spell this out here.

However, we wish to say that the isometry group of $\mathbb{H}$ is given by $\mathrm{PSL}_2(\mathbb{R})$, i.e. the quotient of $\mathrm{SL}_2(\mathbb{R})$ by the subgroup C_2 generated by the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

The quotient topology endows $\mathrm{PSL}_2(\mathbb{R})$ with a topology making it a topological group (in fact, a Lie group). The action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$ is given by what are called Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

Any discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$, that is a subgroup whose subspace topology is the discrete topology, therefore still acts on $\mathbb{H}$ and one can show that this action is proper in the sense of Exercise 3 Sheet 8 [Kha12, Theorem 3.17]. In particular, it is covering-like if it is free. Discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ have a name, they are called *Fuchsian* groups. Consequently, whenever a Fuchsian group Γ acts freely on $\mathbb{H}$ (this turns out to be equivalent to the condition that Γ is torsionfree), we obtain a topological space $\mathbb{H}/\Gamma$ with a covering map $\mathbb{H} \to \mathbb{H}/\Gamma$[3]. The quotient spaces $\mathbb{H}/\Gamma$ then have fundamental group isomorphic to Γ and all higher homotopy groups vanish (since $\mathbb{H}$ is contractible). Many interesting groups arise in this form (this is, however, not obvious from what we have discussed so far): For instance the surface groups $\pi_1(\Sigma_g)$ for $g \geq 2$ are of this kind, as well as the free groups $\mathcal{F}_n$ for $n \geq 1$.

3.19. **Remark** The above type of example is not restricted to 2-dimensional hyperbolic space. For any $n \geq 2$ one can consider n-dimensional hyperbolic space $\mathbb{H}^n$, euclidean space $\mathbb{R}^n$ and for the spheres S^n. These are examples of what are called *Riemannian manifolds of constant curvature*, the curvatures being -1 in the hyperbolic, 0 in the euclidean and 1 in the spherical case. Any discrete subgroup Γ of such isometry groups acting covering-like on these spaces therefore give rise to covering maps, whose quotients have fundamental group Γ, and whose higher homotopy groups are trivial (in case of $\mathbb{H}^n$ and $\mathbb{R}^n$) or that of S^n in the remaining case.

3.20. **Example** We deliver here on the promise to show that $\bigvee_S S^1$ has no higher homotopy groups. Indeed, to show this, it suffices to construct a covering of $\bigvee_S S^1$ whose higher homotopy groups vanish. An appropriate tree does covers $\bigvee_S S^1$, as we work out on Exercise Sheet 9. We may then use that trees are contractible to deduce the result. Using such tricks to calculate fundamental groups is a prominent tool in geometric group theory, and the above ideas are vastly generalized by what is called Bass–Serre theory.

We continue towards describing the category $\mathrm{Cov}(B)$. First, we note that path lifting for coverings gives the following construction.

3.21. **Notation** For a covering map $p\colon E \to B$, and a path γ from b to b' in B, we obtain a map $\tau_\gamma\colon \mathrm{Fib}_b \to \mathrm{Fib}_{b'}$ by sending e to $\bar{\gamma}_e(1)$. We call τ_γ the monodromy action on γ.

[3]The quotient spaces $\mathbb{H}/\Gamma$ therefore locally look homeomorphic to 2-dimensional euclidean space and are consequently examples of 2-dimensional manifolds. For a general Fuchsian group Γ the topological spaces $\mathbb{H}/\Gamma$ are still relatively nice: They are manifolds away from isolated points of singularities and the singularities can be explicitly described in terms of the isotropy groups which appear in the Γ-action on $\mathbb{H}$. These quotient spaces $\mathbb{H}/\Gamma$ are called orbifolds.

3.22. **Lemma** *Let $p\colon E \to B$ be a covering map. The association sending $b \in B$ to $\tau_b := \mathrm{Fib}_b^p = p^{-1}(b)$ and a path γ from b to b' to its monodromy action $\tau_\gamma\colon \mathrm{Fib}_b \to \mathrm{Fib}_{b'}$ assembles into a functor $\tau_p\colon \tau_{\leq 1}(B) \to \mathrm{Set}$.*

Proof. In Theorem 3.10 (2), we have shown that τ_γ depends only on the homotopy class rel endpoints of γ, and hence gives a well-defined function on the set of morphisms in $\tau_{\leq 1}(B)$. We need to check that the axioms of a functor are satisfied, that is, that $\tau_{\mathrm{id}_b} = \mathrm{id}_{\tau_b}$ and that $\tau_{\gamma' \star \gamma} = \tau_{\gamma'} \circ \tau_\gamma$, both of which follow from the uniqueness of path-lifting. □

We call the functor τ_p the monodromy functor of p.

3.23. **Lemma** *For $i = 0, 1$, let $p_i\colon E_i \to B$ be a covering map and let $f\colon E_0 \to E_1$ be a morphism in $\mathrm{Cov}(B)$. Then the canonical maps $f\colon \mathrm{Fib}_b^{p_0} \to \mathrm{Fib}_b^{p_1}$ are the components of a natural transformation $\eta_f\colon \tau_p \to \tau_{p'}$.*

Proof. We need to show that for each path γ from b to b', the following diagram commutes.

$$\begin{array}{ccc} p_0^{-1}(b) & \xrightarrow{f} & p_1^{-1}(b) \\ \downarrow{\scriptstyle \tau_\gamma} & & \downarrow{\scriptstyle \tau_\gamma} \\ p_0^{-1}(b') & \xrightarrow{f} & p_1^{-1}(b') \end{array}$$

So let $e \in p_0^{-1}(b)$. Then the top right composite is given by $e \mapsto \bar{\gamma}_{f(e)}(1)$, whereas the lower left composite is given by $e \mapsto f(\bar{\gamma}_e(1))$. Both of these are the evaluation at 1 of a lift of γ in E_1 with starting point $f(e)$, hence they agree by uniqueness of path lifting. □

3.24. **Corollary** *Let B be a space. There is a functor $\mathrm{Fib}\colon \mathrm{Cov}(B) \to \mathrm{Fun}(\tau_{\leq 1}(B), \mathrm{Set})$ sending a covering map p to its monodromy functor τ_p and a map f of coverings to the natural transformation η_f.*

Proof. We need to check that Fib is compatible with identities and composition of maps of coverings. Both follow immediately from the definitions. □

To formulate the fundamental theorem of covering theory, we need one more definition:

3.25. **Definition** A topological space X is called *semi-locally simply-connected* if for every point $x \in X$ there exists an open subset $V \subseteq X$ containing x such that the map $\pi_1(V, x) \to \pi_1(X, x)$ is trivial.

3.26. **Theorem** (Fundamental theorem of covering theory) *Let B be a locally path connected space. Then the functor*

$$\mathrm{Fib}\colon \mathrm{Cov}(B) \to \mathrm{Fun}(\tau_{\leq 1}(B), \mathrm{Set})$$

is fully faithful. It is essentially surjective (and hence an equivalence of categories) if and only if B is semi-locally simply connected.

The proof will be divided into two parts: A reduction to the path connected case (which we explain in the following remark) and then a proof of an equivalent formulation of Theorem 3.26 in this case.

3.27. **Remark** Suppose B is weakly locally connected. Then B is homeomorphic to the coproduct of its components $B' \in \pi(B)$. We note that in this case, B is semi-locally simply connected if and only if all the components are semi-locally simply connected. The inclusions of these components induce a commutative diagram

$$\begin{array}{ccc} \mathrm{Cov}(B) & \longrightarrow & \mathrm{Fun}(\tau_{\leq 1}(B), \mathrm{Set}) \\ \downarrow & & \downarrow \\ \prod_{B' \in \pi(B)} \mathrm{Cov}(B') & \longrightarrow & \prod_{B' \in \pi(B)} \mathrm{Fun}(\tau_{\leq 1}(B'), \mathrm{Set}) \end{array}$$

of which the vertical maps are isomorphisms of categories (Indeed, a covering over a coproduct of spaces is the same datum as a covering over each of the spaces). Consequently, to prove Theorem 3.26, we may assume that B is globally and locally path connected. In this case, the choice of a basepoint $b \in B$ provides an equivalence $\mathrm{Fun}(\tau_{\leq 1}(B), \mathrm{Set}) \simeq \mathrm{Fun}(B\pi_1(B,b), \mathrm{Set})$ and the latter is the category $\pi_1(B,b)$-Set of sets equipped with an action of $\pi_1(B,b)$, see Observation 3.29 for a recollection and some properties about sets equipped with an action of a group. Under this equivalence, the functor Fib is explicitly given as follows. It sends a covering map $p\colon E \to B$ to $\mathrm{Fib}_b^p = p^{-1}(b)$. This set is acted upon by $\pi_1(B,b)$ by lifting loops to paths. A map of covering spaces is then sent to the induced map on fibres which is $\pi_1(B,b)$-equivariant as we have established before. Theorem 3.26 is therefore equivalent to the following special case of it:

3.28. **Theorem** (Fundamental theorem of covering theory, path connected case) *Let B be a locally and globally path connected space and $b \in B$. Then the functor*

$$\mathrm{Fib}_b\colon \mathrm{Cov}(B) \longrightarrow \pi_1(B,b)\text{-Set}, \quad (p\colon E \to B) \mapsto \mathrm{Fib}_b^p$$

is fully faithful, and essentially surjective if and only if B is semi-locally simply connected.

Proof of fully faithfulness. We first note that since B is locally path connected, given any covering $p\colon E \to B$, then E is also locally path connected. In particular E is the disjoint union over its path connected components. Moreover, the restriction of p to any path connected component is again a covering map (Exercise). So E is a disjoint union of covering maps. We first show that Fib_b is faithful. So let $f, f'\colon E \to E'$ be two maps between $p\colon E \to B$ and $p'\colon E' \to B$ such that their restriction to $p^{-1}(b)$ agree. It suffices to show that f, f' agree on each component E_0 of E. As we have just said, $p_0\colon E_0 \to B$ is a covering. Since B is connected, p_0 is surjective, and hence $p^{-1}(b) \cap E_0$ is non-empty. We then find that f, f' restrict to two maps $E_0 \to E'$ such that $p'f = p'f' = p$ and they agree on a non-empty set. By Lemma 3.9, they agree on all of E_0. Since E_0 was an arbitrary connected component, faithfulness follows. Now, let us show that Fib_b is full. So assume given two covering maps $p\colon E \to B$ and $p'\colon E' \to B$ and a $\pi_1(B,b)$-equivariant map $f\colon p^{-1}(b) \to p'^{-1}(b)$. . We aim to use Proposition 3.14 to lift the map $p\colon E \to B$ along $p'\colon E' \to B$. To do so, we pick basepoints $e_i \in p^{-1}(b)$, $i \in \pi_0(E)$ for each component E_i of E. For each such e, we claim that

$$p_*(\pi_1(E,e)) \subseteq p'_*(\pi_1(E', f(e)))$$

so that Proposition 3.14 gives a map $F_i\colon E_i \to E'$ sending e to $f(e)$. Indeed, let $\gamma \in \pi_1(E,e)$. To see that $p(\gamma)$ lies in $p'_*(\pi_1(E', f(e)))$, it suffices to show that a lift of $p(\gamma)$ with start point $f(e)$ is a closed path. By definition, the endpoint of such a lift is given by the action of $p(\gamma)$ (as an element of $\pi_1(B,b)$) on $f(e)$. In formulas, we need to show that $p(\gamma) \cdot f(e) = f(e)$. By equivariance of the map f, we have $p(\gamma) \cdot f(e) = f(p(\gamma) \cdot e) = f(e)$, where the latter equality holds because γ is a lift of $p(\gamma)$ and γ is a loop at e. Together, the maps F_i assemble into a map $F\colon E \to E'$ of coverings and it remains to show that $F_{|p^{-1}(b)} = f$. By construction, this is true for all $e_i \in p^{-1}(B)$. In general, an element $e \in p^{-1}(B)$ lies in some component E_i of E. Then e and e_i lie in the same component. They are therefore connected by a path γ in E. Then we find $e = \gamma \cdot e_i$. Hence,

$$f(e) = f(\gamma \cdot e_i) = \gamma \cdot f(e_i) = \gamma \cdot F(e_i) = F(\gamma \cdot e_i) = F(e)$$

where we have used that F induces a $\pi_1(B, b)$-equivariant map on $p^{-1}(b)$. This finishes the proof of fully faithfulness. □

We now aim towards proving that the functor Fib_b is essentially surjective.

3.29. **Observation** Let G be a group, for instance $G = \pi_1(B, b)$. Recall that a G-set is a set M equipped with an action of G, i.e. a map $G \times M \to M$, $(g, m) \mapsto gm$, such that $em = m$ for all $m \in M$ and $g(hm) = (gh)m$ for all $g, h \in G$ and all $m \in M$. Equivalently phrased, that the map $G \to \mathrm{Hom}_{\mathrm{Set}}(M, M)$, $g \mapsto (m \mapsto gm)$ is a monoid homomorphism. A map of G-sets (also called an equivariant map) is a map $f\colon M \to M'$ such that $f(gm) = gf(m)$ for all $g \in G$ and all $m \in M$. It is a good exercise to show that the category of G-sets is indeed isomorphic to the functor category $\mathrm{Fun}(BG, \mathrm{Set})$, where BG is the category with one object $*$ and $\mathrm{Hom}_{BG}(*, *) = G$.

Now we recall that any G-set M is a disjoint union of transitive G-sets. To do so, we first recall that a G-set M is called *transitive* if for some (and hence any) $m \in M$, we have that the induced map $G \to M$, given by $g \mapsto g \cdot m$ is surjective. In this case, M is isomorphic to $G/\mathrm{Stab}_G(m)$ where $\mathrm{Stab}_G(m) = \{g \in G \mid g \cdot m = m\}$ is the stabilizer subgroup at m of the action of G on M. Hence, transitive G-sets are non-canonically isomorphic to orbits spaces G/H, for $H \subseteq G$ a subgroup.

Now for a general G-set M, denote by M_m the G-orbit of m, i.e. the image of the map $G \to M$, $g \mapsto gm$. Then clearly $M = \bigcup_{m \in M} M_m$ and M_m is transitive for each $m \in M$. The decisive property is then the following: If $M_m \cap M_{m'} \neq \emptyset$ then $M_m = M'_m$. Indeed, suppose $x \in M_m \cap M_{m'}$. Then $x = gm = g'm'$ for some $g, g' \in M$. But then $m' = g'^{-1}gm \in M_m$. It follows that $M_{m'} \subseteq M_m$ and by symmetry that also $M_m \subseteq M_{m'}$. Consequently, M is the disjoint union of subsets of the form M_m.

3.30. **Corollary** *The functor* $\mathrm{Fib}_b\colon \mathrm{Cov}(B) \to \pi_1(B, b)\text{-Set}$ *is essentially surjective if and only if for all subgroups* $H \subseteq \pi_1(B, b)$ *the orbit space* $\pi_1(B, b)/H$ *is in the image of* Fib_b.

Proof. The only if part is clear. To see the if part, let M be a $\pi_1(B, b)$-set. By Observation 3.29 we can find a set I and subgroups $H_i \subseteq \pi_1(B, b)$ for $i \in I$ such that M is isomorphic to $\coprod_{i \in I} \pi_1(B, b)/H_i$. Pick $p_i\colon E_i \to B$ such that $\mathrm{Fib}_b^{p_i} \cong \pi_1(B, b)/H_i$. Then $p = \coprod_{i \in I} p_i\colon E = \coprod_{i \in I} E_i \to B$ is a covering map with $\mathrm{Fib}_b^p = \coprod \mathrm{Fib}_b^{p_i} \cong M$ as needed. □

We characterize the stabilizers appearing here in terms of the covering space:

3.31. **Lemma** *Let* $p\colon E \to B$ *be a covering map with* E *path connected. Let* $b \in B$ *and let* $e \in p^{-1}(b)$. *Then the canonical map* $p_*\colon \pi_1(E, e) \to \pi_1(B, b)$ *induces an isomorphism* $\pi_1(E, e) \cong \mathrm{Stab}_{\pi_1(B,b)}(e) \subseteq \pi_1(B, b)$. *In particular,* $p^{-1}(b) \cong \pi_1(B, b)/\pi_1(E, e)$ *as* $\pi_1(B, b)$-*set.*

Proof. The first part follows readily from the previously established fact that p_* is injective with image the loops which lift to closed loops at e in E. For the "in particular", note that since E is path connected, we find that $p^{-1}(b)$ is a transitive $\pi_1(B,b)$-set. The claim then follows from Observation 3.29. □

3.32. **Corollary** *Let B be locally and globally path connected space. The $\pi_1(B,b)$ set $\pi_1(B,b)$ is in the image of the functor* Fib_b *if and only if there exists a simply connected covering $\pi\colon \widetilde{B} \to B$.*

Before continuing to address the essential surjectivity of the functor Fib_b, we introduce the following terminology.

3.33. **Definition** An automorphism of a covering $p\colon E \to B$ is called a *Deck-transformation* of p. We therefore write $\mathrm{Deck}(p) = \mathrm{Aut}_{\mathrm{Cov}(B)}(p)$. For any choice of basepoint $b \in B$, the group $\mathrm{Deck}(p)$ acts on $p^{-1}(b)$. A covering map $p\colon E \to B$ is called a *Galois covering* or a *normal covering* if for all $b \in B$, the action of $\mathrm{Deck}(p)$ on $p^{-1}(b)$ is transitive.

3.34. **Remark** Let $p\colon E \to B$ be a covering map and let $e \in p^{-1}(b)$ be a basepoint of E. Assume that E is connected. Then the action of $\mathrm{Deck}(p)$ on $p^{-1}(b)$ is free, i.e. the induced map $\mathrm{Deck}(p) \to p^{-1}(b)$ given by $f \mapsto f(e)$ is injective. Indeed, suppose given $f, f' \in \mathrm{Deck}(p)$ such that $fe = f'e$. Then Lemma 3.9 shows that $f = f'$. Consequently, for connected E, $p\colon E \to B$ is a Galois covering if and only if the action of $\mathrm{Deck}(p)$ on $p^{-1}(b)$ is free and transitive, or again equivalently, if the map $\mathrm{Deck}(p) \to p^{-1}(b)$, $f \mapsto f(e)$ is bijective for all $e \in p^{-1}(b)$.

Considering for any covering map $p\colon E \to B$ the canonical covering map $p \amalg p\colon E \amalg E \to B$ we find the following. Namely we have $\mathrm{Deck}(p) \times \mathrm{Deck}(p) \subseteq \mathrm{Deck}(p \amalg p)$ induced by sending (f,g) to $f \amalg g\colon E \amalg E \to E \amalg E$. The subgroup $\mathrm{Deck}(p) \times 1 \subseteq \mathrm{Deck}(p) \times \mathrm{Deck}(p) \subseteq \mathrm{Deck}(p \amalg p)$ fixes all points in the second copy of E. Hence, if $\mathrm{Deck}(p) \neq 1$, we find that the action of $\mathrm{Deck}(p \amalg p)$ on E is not free. Therefore, in general, we cannot drop the assumption that E is connected in the above argurment.

3.35. **Lemma** *Let $p\colon E \to B$ be a covering map with E connected. Then the action of* $\mathrm{Deck}(p)$ *on E is covering-like.*

Proof. Let $e \in E$ and pick an open $U \subseteq E$ containing e such that p restricts to a homeomorphism on U. Let $f \in \mathrm{Deck}(p)$ and assume that $x \in f(U) \cap U$. Then there exists $y \in U$ such that $fy = x$. In particular, $py = pfy = px$ since $pf = p$. Since p is injective on U, we deduce that $x = y$ and therefore that $fx = x$. But we have just recorded that the action of $\mathrm{Deck}(p)$ on E is free, so $f = \mathrm{id}_E$ and $\mathrm{Deck}(p)$ acts covering-like on E. □

3.36. **Example** Let $p\colon E \to B$ be a covering map with E connected. Then the tautological action of $\mathrm{Deck}(p)$ on E is covering-like as we have just argued. Moreover, the map $p\colon E \to B$ factors through the quotient map

$$E \longrightarrow E/\mathrm{Deck}(p) \longrightarrow B$$

of which the first map and the composite are covering maps. This implies that $E/\mathrm{Deck}(p) \to B$ a covering map (we leave this as an exercise, but note that this statement uses that $E \to E/\mathrm{Deck}(p)$ is surjective). Now let B be locally and globally path-connected. Then $p\colon E \to B$ is a Galois covering if and only if the induced map $E/\mathrm{Deck}(p) \to B$ is a homeomorphism. Indeed, to show this, it suffices to show that the map on fibres over points in b is an bijection if and only if $p\colon E \to B$ is Galois. Now, by construction, the fibres of B are one point sets.

The fibre of $E/\mathrm{Deck}(p)$ over $b \in B$ is given by $p^{-1}(b)/\mathrm{Deck}(p)$. This is a singleton if and only if the action is transitive, i.e. if and only if E is Galois.

3.37. **Example** Let B be a locally and globally path-connected space and let $p\colon E \to B$ be a covering map with E connected. For any $e \in E$ we obtain an isomorphism $\mathrm{Deck}(p) \cong N_{\pi_1(B,p(e))}(\pi_1(E,e))/\pi_1(E,e))$. Indeed, in general for a group G and a subgroup H of G, we obtain an isomorphism $\mathrm{Aut}_G(G/H) = N_G(H)/H$. Here, $N_G(H)$ denotes the normalizer of H in G, i.e. the largest subgroup of G which contains H as a normal subgroup, concretely given by $N_G(H) = \{g \in G \mid gH = Hg\}$.

The following corollary explains why Galois coverings are also called normal coverings.

3.38. **Corollary** *Let B be locally and globally path connected and $p\colon E \to B$ a connected covering. Then E is Galois if and only if for any $e \in E$, the characteristic subgroup $p_*(\pi_1(E,e)) \subseteq \pi_1(B,b)$ is a normal subgroup.*

Proof. Let $b = p(e)$. By Lemma 3.31, we have $p^{-1}(b) \cong \pi_1(B,b)/\pi_1(E,e)$ as a $\pi_1(B,b)$-set. By definition, E is Galois if the action map $\mathrm{Deck}(p) \to p^{-1}(b)$, $f \mapsto f(e)$ is surjective. By Example 3.37 we have that $\mathrm{Deck}(p) \cong N_{\pi_1(B,b)}(\pi_1(E,e))/\pi_1(E,e)$. Under this isomorphism, and the isomorphism $p^{-1}(b) \cong \pi_1(B,b)/\pi_1(E,e)$, the action map is given by the map $N_{\pi_1(B,b)}(\pi_1(E,e))/\pi_1(E,e) \to \pi_1(B,b)/\pi_1(E,e)$ induced by the inclusion (that this is so of course requires some unravelling of definitions). This map is surjective if and only if the inclusion $N_{\pi_1(B,b)}(\pi_1(E,e)) \subseteq \pi_1(B,b)$ is surjective, i.e. precisely when $\pi_1(E,e)$ is normal. Since e was arbitrary the lemma follows. □

3.39. **Remark** One can also give a direct proof of Corollary 3.38 as follows. Let $e, e' \in p^{-1}(b)$ and let γ be a path from e to e' in E. Then $p(\gamma) \in \pi_1(B,b)$ and we have

$$p_*(\pi_1(E,e)) = p(\gamma)^{-1} \cdot p_*(\pi_1(E,e')) \cdot p(\gamma).$$

Hence, if all characteristic subgroups are normal, we may use Proposition 3.14 to obtain a map $f\colon E \to E$ in $\mathrm{Cov}(B)$ sending e to e'. Likewise, we can find $f'\colon E \to E'$ sending e' to e. Both composites ff' and $f'f$ are then, by uniqueness, the identity of E so that $f \in \mathrm{Deck}(p)$. Consequently, $\mathrm{Deck}(p)$ acts transitively on $p^{-1}(b)$. Conversely, suppose that $\mathrm{Deck}(f)$ acts transitively on $p^{-1}(b)$. Pick $\alpha \in \pi_1(B,b)$ and $e \in p^{-1}(b)$. It suffices to show that

$$p_*(\pi_1(E,e)) = \alpha \cdot p_*(\pi_1(E,e)) \cdot \alpha^{-1}.$$

Let $\bar{\alpha}$ be a lift of α with start point e and let e' be its endpoint. Then as observed before, we have $\alpha \cdot p_*(\pi_1(E,e)) \cdot \alpha^{-1} = p_*(\pi_1(E,e'))$. Now let $f \in \mathrm{Deck}(p)$ be such that $f(e') = e$. Then $p_*(\pi_1(E,e')) = p_*(f_*(\pi_1(E,e)) = p_*(\pi_1(E,e))$ since f_* is an isomorphism.

We continue with reduction steps towards essential surjectivity of the fibre functor.

3.40. **Corollary** *Let B be a locally and globally path connected space and $b \in B$. Then the functor* $\mathrm{Fib}_b\colon \mathrm{Cov}(B) \to \pi_1(B,b)$-Set *is essentially surjective if and only if there exists a simply connected covering of B.*

Proof. If Fib_b is essentially surjective, then there exists a connected covering $p\colon E \to B$ with $\mathrm{Fib}_b^p \cong \pi_1(B,b)$ and hence with $\pi_1(E,e) = 1$. Conversely, if $\pi\colon \widetilde{B} \to B$ is a simply connected covering, then π is a Galois covering by Corollary 3.38 and $\mathrm{Deck}(\pi) \cong \pi_1(B,b)$ by Example 3.37. Since $\mathrm{Deck}(\pi)$ acts covering-like on $\widetilde{B}$ by Lemma 3.35, any subgroup

$H \subseteq \mathrm{Deck}(\pi) \cong \pi_1(B,b)$ also acts covering-like on $\widetilde{B}$. Hence, for any subgroup $H \subseteq \pi_1(B,b)$, we may form the quotient $\widetilde{B}/H$ whose canonical map to B is again a covering map. By construction, the fibres of this quotient map are isomorphic to $\pi_1(B,b)/H$. □

To finish the proof of the fundamental theorem of covering theory we therefore only need to show the following result:

3.41. **Proposition** *Let B be a locally and globally path connected space. Then there exists a simply connected covering $\pi\colon \widetilde{B} \to B$ if and only if B is semi-locally simply connected.*

Proof. Let $\pi\colon \widetilde{B} \to B$ be a simply connected covering. Let $b \in B$ and let $U \subseteq X$ be a trivializing open subset containing x. Consider $\gamma \in \pi_1(U,x)$. Since π is trivial on U, we can lift γ to a loop $\gamma'\colon S^1 \to \pi^{-1}(U)$. Moreover, the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \longrightarrow & \widetilde{B} \\ \downarrow & & \downarrow \\ U & \longrightarrow & B \end{array}$$

commutes. Since $\pi_1(\widetilde{B},\tilde{b})$ is trivial for all $\tilde{b}$ we see that the map $\pi_1(U,b) \to \pi_1(B,b)$ is trivial, so B is semi-locally simply connected.

The converse is of course more elaborate: We need to construct a simply connected covering. We define $\widetilde{B}$ as follows. Its underlying set is given by

$$\{\gamma\colon [0,1] \to B \mid \gamma(0) = b\}/\text{homotopy rel endpoints} = \coprod_{x \in B} \mathrm{Hom}_{\tau_{\leq 1}(B)}(b,x)$$

and the projection map $\widetilde{B} \to B$ is given by $[\gamma] \mapsto \gamma(1)$. We topologize $\widetilde{B}$ as follows. A subset $U \subseteq \widetilde{B}$ is open if for each $[\gamma] \in U$, there exists a path connected open $V \subseteq B$ containing $\gamma(1)$ such that for every path $\gamma'\colon [0,1] \to V$ with $\gamma'(0) = \gamma(1)$, we have $[\gamma' \star \gamma] \in U$. This is indeed a topology on $\widetilde{B}$: Obviously $\varnothing$ is open, and $\widetilde{B}$ is also open. Indeed, for the required path connected neighborhoods of $\gamma(1)$ we can choose B since B is globally path connected. That the union of opens is open is tautologically true. To see that finite (equivalently by induction binary) intersections of opens are open, one uses the local path connectivity of B: Suppose $[\gamma] \in U_1 \cap U_2$ with U_1, U_2 open. Choose V_1, V_2 path connected neighborhoods of U_1 and U_2 as in the definition of the open sets. Then choose a path connected open $V \subseteq V_1 \cap V_2$ containing $\gamma(1)$. It has the desired properties.

To see that π is continuous, let $U \subseteq B$ open. To see that $\pi^{-1}(U)$ is open, pick $[\gamma] \in \pi^{-1}(U)$. By local path connectivity of B, we may choose a path connected open subset $V \subseteq U$ with $\gamma(1) \in V$. Consequently, for all $\gamma'\colon [0,1] \to V$ with $\gamma'(0) = \gamma(1)$, we find $[\gamma' \star \gamma] \in \pi^{-1}(U)$, so $\pi^{-1}(U)$ is open.

Next, we wish to show that $\pi\colon \widetilde{B} \to B$ is a covering map. To do so, consider a point $b' \in B$ and an open path connected subset V containing b' with the property that $\pi_1(V,b') \to \pi_1(B,b')$ is trivial. This can be done since B is locally path connected and semi-locally simply connected. We will construct a homeomorphism $\pi^{-1}(V) \to V \times F$ for a discrete set F, compatible with the projections to V. So first, we need to construct F and a continuous map $\pi^{-1}(V) \to F^{\delta}$. We let $F = \mathrm{Hom}_{\tau_{\leq 1}(B)}(b,b')$. Now let $[\gamma] \in \pi^{-1}(V)$, i.e. $\gamma(1) \in V$. Choose an arbitrary path α from $\gamma(1)$ to b' inside V. Then define $\theta([\gamma]) = [\alpha \star \gamma] \in \mathrm{Hom}_{\tau_{\leq 1}(B)}(b,b')$. First, we note that this map is independent of the choice of α: Indeed, since $\alpha'^{-1} \star \alpha \in \pi_1(V,b)$, its image in $\pi_1(B,b)$ is trivial. Hence, the paths $\alpha' \star \gamma$ and $\alpha \star \gamma$ represent the same element in $\mathrm{Hom}_{\tau_{\leq 1}(B)}(b,b')$. We claim that the resulting map $\pi^{-1}(V) \to V \times F^\delta$ is a homeomorphism.

To see that it is bijective, we note that an inverse is given by sending the pair (v,γ) to $[\alpha \star \gamma]$ where again α is a path from b' to v in V. It remains to show that the so constructed bijection $\pi^{-1}(V) \to V \times F^\delta$ is continuous and open. To see continuity, consider $V' \subseteq V$ open, and $[\bar{\gamma}] \in F$. Then

$$(\pi \times \theta)^{-1}(V' \times \{\bar{\gamma}\}) = \{[\gamma] \mid \gamma(1) \in V', [\alpha \star \gamma] = [\bar{\gamma}]\}$$

where α is a path in V from $\gamma(1)$ to x. To see that this is open, consider an element $[\gamma]$ in it. Choose a path connected open subset $\bar{V} \subseteq V'$ and let β be a path from $\gamma(1)$ to $v' \in V'$. To see that $[\beta \star \gamma] \in (\pi \times \theta)^{-1}(V' \times \{[\bar{\gamma}]\})$ (and hence that the latter set is open) note that $\pi(\beta \star \gamma) = \beta(1) \in V' \subseteq V$. Moreover, to calculate $\theta([\beta \star \gamma])$, we may choose an arbitrary path from $\beta(1)$ to x inside V. First, pick α a path from $\gamma(1)$ to x in V. Then consider $\alpha \star \beta^{-1}$, which is a path from $\beta(1)$ to x inside V, showing that $\theta([\beta \star \gamma]) = [\alpha \star \gamma] = [\bar{\gamma}]$ as needed. It follows that $\pi^{-1}(V)$ is given by the coproduct of the subsets $(\pi \times \theta)^{-1}(V \times \{f\})$. Consequently, it suffices to show that the restricted maps $(\pi \times \theta)\colon (\pi \times \theta)^{-1}(V \times \{f\}) \to V \times \{f\} \cong V$ are open. So let $U \subseteq (\pi \times \theta)^{-1}(V \times \{f\})$ be open. For $[\gamma] \in U$, there is then a path connected open subset $W \subseteq V$ such that for any path β from $\gamma(1)$ to w, we have that $\beta \star \gamma \in U$. This implies that $W \subseteq \pi(U)$, so that $\pi(U)$ is open by the self-indexing trick.

Finally, we need to show that $\widetilde{B}$ is path connected and simply connected. So consider an element $[\gamma] \in \widetilde{B}$. We consider the map $\Phi_\gamma\colon [0,1] \to \widetilde{B}$ sending s to $[\gamma(s \cdot -)]$. To see that it is continuous, let $U \subseteq \widetilde{B}$ be open with $\Phi_\gamma(s) \in U$. This means that there exists a path connected open subset $V \subseteq B$ containing $\gamma(s)$ such that for all $\alpha\colon [0,1] \to V$ with $\alpha(0) = \gamma(s)$, we have $[\alpha \star \gamma(s-)] \in U$. Since $\gamma\colon [0,1] \to B$ is continuous and $\gamma(s) \in V$, we find that $\gamma(t) \in V$ for $|t-s| < \epsilon$ for some ϵ. Now notice that $[\gamma(t \cdot -)] = [\gamma(s,t) \star \gamma(s \cdot -)]$ where $\gamma(s,t)$ is the path γ restricted to the interval $[s,t]$ when $t \geq s$ and its inverse restricted to $[t,s]$ when $t < s$. We deduce that $[\gamma(t \cdot -)] \in U$ for $|t-s| < \epsilon$ and hence that Φ_γ is continuous. Moreover, it is a path in $\widetilde{B}$ from $[\mathrm{const}_b]$ to $[\gamma]$, so $\widetilde{B}$ is path-connected.

Likewise, let $\gamma\colon [0,1] \to \widetilde{B}$ be a closed path at $[\mathrm{const}_b]$. By Corollary 3.11, it suffices to show that $\pi \circ \gamma$ is homotopic to the constant path at b. Now, $\pi \circ \gamma$ is the path sending $t \in [0,1]$ to $\gamma(t)(1)$. It is therefore homotopic rel endpoints to the path sending t to $\gamma(t)(0)$, which, by construction of $\widetilde{B}$ is given by the constant path at b. □

We now briefly mention what we like to refer to as a universal cover (rather than a simply connected cover):

3.42. **Definition** Let B be a space. For $b \in B$ we may consider the functor

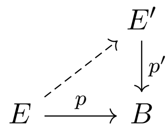
$$\mathrm{Fib}_b\colon \mathrm{Cov}(B) \longrightarrow \mathrm{Set}$$

We say that B admits a universal cover at b if Fib_b is representable, and call any representing object a universal cover at b.

Unravelling the definitions, a universal cover at b is therefore a covering map $p\colon E \to B$ equipped with an isomorphism $\mathrm{Hom}_{\mathrm{Cov}(B)}(E,-) \simeq \mathrm{Fib}_b$. Such an isomorphism in particular determines (by Yoneda) an element e in Fib_b^p. That the resulting natural transformation $\tau_e\colon \mathrm{Hom}_{\mathrm{Cov}(B)}(E,-) \to \mathrm{Fib}_b$ is an isomorphism concretely means that for any other covering map $p'\colon E' \to B$ and any element $e' \in \mathrm{Fib}_b^{p'}$, there exists a unique map of coverings $f\colon E \to E'$ with $f(e) = e'$.

3.43. **Lemma** *Let $p\colon E \to B$ be a covering map with E simply connected and locally and globally path connected. Then $p\colon E \to B$ is universal at b for all $b \in p(E)$.*

Proof. Let $e \in p^{-1}(b)$ and let $p'\colon E' \to B$ be another covering and $e' \in p'^{-1}(b)$. Consider the following diagram



Then by Proposition 3.14 a dashed arrow sending e to e' exists (uniquely) if and only if $p_*(\pi_1(E,e)) \subseteq p'_*(\pi_1(E',e'))$. This is tautologically true since $\pi_1(E,e)$ is trivial. $\square$

In other words, for appropriate B (e.g. locally and globally path connected), any simply connected covering is a universal covering at all points of B. The converse is not true. Thinking about precise relationships is a nice topic for a bachelor's thesis. However, for locally and globally path connected, as well as semi-locally simply connected spaces, the notion of universal and simply connected coverings agree.

Finally, we record the following characterizations of universality:

3.44. **Lemma** *Let B be locally and globally path connected and $p\colon E \to B$ a covering map. Then the following are equivalent:*

(1) *p is universal at all points $b \in B$,*
(2) *p is universal at some $b \in B$, and*
(3) *E is connected and Galois and for any covering map $p'\colon E' \to B$, there exists a map of coverings $E \to E'$.*

Proof. Clearly, (1) implies (2). Let us now assume (2). First, we show that E is connected. So let $e \in p^{-1}(b)$ and let E_e be the component of e. Then $p_e\colon E_e \to B$ is a covering map. Consequently, there exists a unique map $E \to E_e$ sending e to e. The composite $E_e \to E \to E_e$ is then a self-map of a connected covering map which fixes the point e. By Lemma 3.9, it is the identity. Likewise, the composite $E \to E_e \to E$ is a self-map of E which fixes the point e. By universality of p at b, it is again the identity. In particular, E is homeomorphic to E_e and hence connected. Now we show that E is Galois. Again, universality implies that $\mathrm{Deck}(p)$ acts transitively on $p^{-1}(b)$. Therefore, for any point $e \in p^{-1}(b)$, the subgroup $p_*(\pi_1(E,e))$ is normal in $\pi_1(B,b)$. We need to show that for any $e' \in E$, the subgroup $p_*(\pi_1(E,e')) \subseteq \pi_1(B,p(e'))$ is normal. To see this, we pick a path γ from e to e'. This induces an isomorphism $\pi_1(E,e') \cong \pi_1(E,e)$. Moreover, $p(\gamma)$ induces an isomorphism $\pi_1(B,p(e)) \cong \pi_1(B,p(e'))$. Under these isomorphisms $p_*(\pi_1(E,e')) \subseteq \pi_1(B,p(e'))$ corresponds to $p_*(\pi_1(E,e)) \subseteq \pi_1(B,p(e))$, so is again normal. Now let us consider a covering map $p'\colon E' \to B$. Then $p'^{-1}(b)$ is non-empty, so we may choose e' in it. By universality of E at b there is a unique map $E \to E'$ of coverings sending e to e'. In particular, there exists a map of coverings from E to E'. Therefore, (2) implies (3). Now assume (3). To show that E is universal at $b \in B$, it suffices to show that for all $e \in p^{-1}(b)$, all coverings $p'\colon E' \to B$ and $e' \in p'^{-1}(b)$, there exists a map $f\colon E \to E'$ with $f(e) = e'$ (by Lemma 3.9 and connectedness of E, such a map is unique). We may assume that E' is also connected since the inclusion of any connected component of E' is a map of coverings. By assumption, there is a map

$g\colon E \to E'$ of coverings. Pick a path γ from $g(e)$ to e' and left $\bar{\gamma}$ be a lift of $p'(\gamma)$ along p with startpoint e. Then $\bar{e} = \bar{\gamma}(1) \in p^{-1}(b)$ and $g(\bar{e}) = e'$. Since E is Galois, there exists a Deck transformation φ of E sending e to $\bar{e}$. The composite $f = g \circ \varphi$ is then a map $E \to E'$ sending e to e'. □

The following is a nice and purely group theoretic application of all we have seen so far.

3.45. **Theorem** *Any subgroup of a free group is free.*

Proof. Let $H \subseteq F$ be a subgroup of a free group F. Choose a generating set S for F, so that there is an isomorphism $\mathcal{F}_S \cong F$. We now show that $\mathcal{F}_S \cong \pi_1(\bigvee_S S^1)$. Indeed, we have already seen this in case S is finite using Seifert van Kampen, see Example 2.68 and 2.70. In general, we claim that the canonical map

$$\operatorname*{colim}_{S' \subseteq S} \bigvee_{S'} S^1 \to \bigvee_S S^1$$

where S' runs through the finite subsets of S is a homeomorphism. To see this, we consider the pushout of the diagram

$$\operatorname*{colim}_{S' \subseteq S} \textstyle\coprod_{S'} D^1 \longleftarrow \operatorname*{colim}_{S' \subseteq S} \coprod_{S'} S^0 \longrightarrow \operatorname*{colim}_{S' \subset S} *$$

Since colimits commute with each other and since $\operatorname{colim}'_S \coprod_{S'} X = \coprod_S X$ for any topological space, as well as $\operatorname{colim}'_S * = *$, we obtain the claimed homeomorphism

$$\operatorname*{colim}_{S' \subseteq S} \bigvee_{S'} S^1 \cong \bigvee_S S^1.$$

We now use that the canonical map

$$\operatorname*{colim}_{S' \subseteq S} \pi_1(\bigvee_{S'} S^1) \to \pi_1(\operatorname*{colim}_{S' \subseteq S} \bigvee_{S'} S^1) \cong \pi_1(\bigvee_S S^1)$$

is an isomorphism. This is not trivial (in the sense that in general $\pi_1(-)$ does not commute with arbitrary filtered colimits) and follows essentially from the fact that S^1 is compact, so that the image of any continuous map $S^1 \to \bigvee_S S^1$ is contained in $\bigvee_{S'} S^1$ for some finite subset $S' \subseteq S$, see Proposition 2.35 (1). Hence, we obtain an isomorphism

$$\mathcal{F}_S \cong \operatorname*{colim}_{S' \subseteq S} \mathcal{F}_{S'} \cong \pi_1(\bigvee_S S^1).$$

The first isomorphism holds since the functor $S' \mapsto \mathcal{F}_{S'}$ commutes with colimits (it is a left adjoint) and $\operatorname{colim}_{S' \subseteq S} S' = S$.

We observe that $\bigvee_S S^1$ is locally and globally path connected and semi-locally simply connected, in fact, every point has a contractible neighborhood, see Proposition 2.35 (6). In particular, by Theorem 3.28 there exists a covering space $X \to \bigvee_S S^1$ such that $\pi_1(X, x) \cong H$. Now on Exercise Sheet 11 we show that a covering space of a 1-dimensional CW complex (such as $\bigvee_S S^1$) again admits the structure of a 1-dimensional CW complex. In Exercise 3 Sheet 6, we have shown that any 1-dimensional CW complex is homotopy equivalent to $\bigvee_T S^1$ for some set T. Hence, by the previous reasoning, we have $H \cong \pi_1(\bigvee_T S^1) \cong \mathcal{F}_T$, so H is a free group. □

4. Singular homology

4.1. **A first glance at homology.** We have now seen that understanding the fundamental group and covering theory allowed us to deduce some nice consequences, e.g. that subgroups of free groups are free and that S^1 is not homotopy equivalent to S^n for $n \geq 2$. In particular, we can extend Proposition 1.50 to one further case, namely that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^k$ for $k \neq 2$. Indeed, if $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^n$, then $S^1 \simeq \mathbb{R}^2 \setminus \{0\} \cong \mathbb{R}^n \setminus \{x\} \simeq S^{n-1}$. In particular, we get $n = 2$. The main geometric reason here is that $\pi_1(-)$ is able to see that S^1 has a hole, but S^n for $n \geq 2$ does not have a hole. In some sense, however, S^n has an n-dimensional hole and we would like to have an invariant which detects that this is so. We have already indicated that $\pi_n(-)$ in fact does the job (though actually proving this requires some work), but it would be nice to have an invariant which sees that S^n has precisely one n-dimensional hole an no holes of other dimension. It will turn out that *singular homology* is good invariant which does this. This invariant is extremely useful for many other purposes, too, so it is time that we introduce it. Before giving a definition of singular homology and establishing its properties, let us list some desirable properties of homology.

Any form of homology should consist of abelian group valued functors $X \mapsto H_n(X)$ for all $n \geq 0$, called the *homology groups* of a topological space X. These functors should satisfy a number of properties:

(1) They should be non-trivial but as easy as possible. In particular, they should satisfy $H_n(\emptyset) = 0$ for all $n \geq 0$ and $H_0(*) = \mathbb{Z}$ and $H_n(*) = 0$ for $n > 0$.
(2) They should be homotopy invariant in the sense of Remark 2.8, that is if $f, g \colon X \to Y$ are homotopic maps, then $H_n(f) = H_n(g)$ as maps $H_n(X) \to H_n(Y)$.
(3) They should be additive in the sense that $H_n(X) \oplus H_n(Y) \cong H_n(X \amalg Y)$[4] for all $n \geq 0$ and $H_n(X) \oplus H_n(Y) \cong H_n(X \vee Y)$ for $n > 0$.
(4) They should detect n-dimensional holes in that its values on spheres are given by

$$H_n(S^k) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0, k \\ 0 & \text{else} \end{cases}$$

4.1. **Remark** It is worthwile to note that the family of functors $\pi_n(-)$ does not give rise to homology in the above sense. First and foremost, it doesn't take values in abelian groups. One could remedy this fact by replacing $\pi_0(X)$ by $\mathbb{Z}[\pi_0(X)]$ and $\pi_1(X)$ by its abelianization $\pi_1(X)^{\mathrm{ab}}$.[5]

But also, $\pi_n(S^k)$ is often non-zero when $n > k$ (this is not at all clear, but a true statement), $\pi_n(-)$ is not additive on disjoint unions of spaces (it sees only the component of a chosen basepoint) and not additive on wedge sums of spaces (again, this is not clear, but a true statement). In addition, $\pi_n(-)$ simply is not a functor on topological spaces (as it requires the choice of a basepoint).

In any case, as a consequence of the above properties (2) and (4) above we obtain a well-defined map

$$\deg \colon [S^n, S^n] \longrightarrow \mathrm{Hom}(H_n(S^n), H_n(S^n)) \xleftarrow{\cong} \mathbb{Z}, \quad [f] \mapsto H_n(f) \leftarrow\!\shortmid \deg(f)$$

[4]In fact, they should send arbitrary coproducts of topological spaces to sums of abelian groups.
[5]It will turn out that singular homology looks precisely like this in degrees 0 and 1.

where $\deg(f)$ is called the *homological degree* of f. By definition, it records the effect of f on the top-dimensional homology of S^n.

We will establish a number of key results about the homological degree below. In order to do so, we will introduce an improved version of (4) which gives not only the same answer, but also a reason for why it should be true. To put it into perspective, recall that there is a canonical homeomorphism $S^n \cong \Sigma(S^{n-1})$. For a pointed space (X, x) let us set $\widetilde{H}_k(X) = \operatorname{coker}(H_k(*) \to H_k(X))$. The improved version of (4) then reads as follows:

(4') For every topological space X and every $n \geq 0$, there is a natural isomorphism

$$\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(\Sigma(X))$$

called the *suspension isomorphism.*

Exercise: Property (4') implies property (4) above.

Property (4') shows that in a homology theory, the functors $H_n(-)$ for different values of n are not unrelated to each other. Later, we will see another form of this relation which is a (vastly) improved version of both (3) and (4) above. With (4') at hand, we can prove that the homological degree has the following properties.

4.2. **Proposition** *Let $n \geq 1$ be a natural number and $f \colon S^n \to S^n$ be a continuous map. The homological degree has the following properties.*

1. *If f is homotopic to a constant map, then* $\deg(f) = 0$ *and* $\deg(\mathrm{id}_{S^1}) = 1$.
2. $\deg(g \circ f) = \deg(g) \cdot \deg(f)$,
3. $\deg(\Sigma(f)) = \deg(f)$
4. $\deg(r) = -1$, *whenever $r \colon S^n \to S^n$ is a reflection along any hyperplane.*
5. $\deg(-\mathrm{id}_{S^n}) = (-1)^{n+1}$.
6. $\deg \colon [S^n, S^n] \to \mathbb{Z}$ *is surjective.*

Proof. (1) follows from the fact that $H_n(-)$ is homotopy invariant, $H_n(*) = 0$ and $H_n(-)$ is a functor. (2) follows from the fact that $H_n(-)$ is a functor. The naturality of the suspension isomorphism gives rise to the following commutative diagram, showing (3).

$$\begin{array}{ccc} [S^n, S^n] & \xrightarrow{\ \Sigma\ } & [S^{n+1}, S^{n+1}] \\ & {\scriptstyle \deg}\searrow \quad \swarrow{\scriptstyle \deg} & \\ & \mathbb{Z} & \end{array}$$

To see (4), note that for any reflection r, there is a isometric homeomorphism $\varphi \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $\varphi r \varphi^{-1}$ is given by the map $(x_1, \dots, x_{n+1}) \mapsto (x_1, -x_2, , x_3 \dots, x_{n+1})$. This map is an $(n-1)$-fold suspension of the map $(x_1, x_2) \mapsto (x_1, -x_2)$, which restricted to S^1 is the map $x \mapsto \bar{x} = x^{-1}$. Hence we deduce from (1)–(3) that

$$\deg(r) = \deg(\varphi) \cdot \deg(r) \cdot \deg(\varphi)^{-1} = \deg(\varphi r \varphi^{-1}) = \deg(x \mapsto x^{-1}) = -1.$$

For (5), simply note that $-\mathrm{id}_{S^n}$ is an $(n+1)$-fold composite of reflections:

$$(x_1, \dots, x_{n+1}) \mapsto (-x_1, x_2, \dots, x_n) \mapsto (-x_1, -x_2, \dots, x_n) \mapsto \dots \mapsto (-x_1, \dots, -x_n).$$

Hence, $\deg(-\mathrm{id}_{S^n}) = (-1)^{n+1}$ by (2). Finally, we show (6). By (3), it suffices to show that $\deg \colon [S^1, S^1] \to \mathbb{Z}$ is bijective. We will show that the map $\deg(x \mapsto x^n) = n$. To do so, we first note that $\deg(\mathrm{id}_{S^1}) = 1$, by functoriality of $H_1(-)$. Now let $f, g \in [S^1, S^1]_* = \pi_1(S^1)$. Recall that $f + g \in \pi_1(S^1)$ is represented by the composite

$$S^1 \xrightarrow{p} S^1 \vee S^1 \xrightarrow{f \vee g} S^1.$$

Applying $H_1(-)$ to this diagram and using that homology is additive on wedge sums, we obtain a diagram

$$\mathbb{Z} \xrightarrow{p_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\deg(f) \oplus \deg(f)} \mathbb{Z}$$

where the first map is the diagonal by the counitality of the pinch map p. Hence the composite is given by $\deg(f) + \deg(g)$ so we obtain the formula

$$\deg(f + g) = \deg(f) + \deg(g).$$

Since $[x \mapsto x^n] = [\mathrm{id}] + \cdots + [\mathrm{id}]$, we find that $\deg(x \mapsto x^n) = n$ for all $n \geq 0$. Now, for $n < 0$, we have $(x \mapsto x^n)$ is the composite of a reflection with $x \mapsto x^{-n}$. The result then follows from (4). □

4.3. **Remark** Let us consider the composite

$$\mathbb{Z} \xrightarrow{\cong} [S^1, S^1]_* \to [S^1, S^1] \xrightarrow{\deg} \mathbb{Z}.$$

Part of Exercise 3 Sheet 11 is to show that this composite is the identity. Since the middle map is surjective (Exercise), we deduce that it is also bijective. Consequently, all maps in the above display are in fact bijective.

Moreover, one can show that the map $\Sigma\colon [S^n, S^n] \to [S^{n+1}, S^{n+1}]$ is surjective for all $n \geq 1$. This is for istance a consequence of Freudenthal's suspension theorem which we will discuss at some point next term. In particular, with this at hand, it follows that the degree map $\deg\colon [S^n, S^n] \to \mathbb{Z}$ is bijective for all $n \geq 1$ (this, a generalization of this result was proven by Hopf and is therefore often referred to as the theorem of Hopf).

4.2. **Applications of homology.** Before explaining how one could try to construct an homology theory in the above sense, we want to give several applications which make only use of the above listed properties. Hopefully, these will motivate us to go through the (rather lengthy and partly technical) construction of homology and the verification of its basic properties (which includes more than the above mentioned properties).

4.4. **Theorem** (Invariance of dimension) *If $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$, then $n = m$.*

Proof. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a homeomorphism. Then f restricts to a homeomorphism $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m \setminus \{f(0)\}$. In particular, f induces a homotopy equivalence $S^{n-1} \simeq S^{m-1}$. Consequently we find that

$$H_{n-1}(S^{m-1}) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}.$$

By the properties of homology alluded to above, this implies that $n = 1$ or $n = m$. If $n = 1$ we may appeal to Proposition 1.50 to see that $m = 1$ as well (recall that $\mathbb{R}$ is homeomorphic to $(0, 1)$). □

4.5. **Theorem** (Brouwer's fixed point theorem) *Let $f\colon D^n \to D^n$ be a continuous function. Then there exists $x \in D^n$ such that $f(x) = x$.*[6]

[6] Such a point is called a fixed point for f.

Proof. Assume that f is such that there is no fixed point. Then we consider a function $g\colon D^n \to S^{n-1}$ given by sending x to the intersection of $\{tx + (1-t)f(x) \mid t \geq \frac{1}{2}\}$ with S^{n-1}. Then g is a continuous function whose restriction to S^{n-1} is the identity. Applying homology we obtain that the composite

$$H_{n-1}(S^{n-1}) \to H_{n-1}(D^n) \xrightarrow{g_*} H_{n-1}(S^{n-1})$$

is the identity (because $H_{n-1}(-)$ is a functor). This is a contradiction: For $n > 1$ the above is isomorphic to the composite $\mathbb{Z} \to 0 \to \mathbb{Z}$ (which is not the identity) and for $n = 1$ it is the composite $\mathbb{Z}^2 \to \mathbb{Z} \to \mathbb{Z}^2$ (which is also not the identity). □

4.6. **Theorem** (The Surjectivity theorem) *Let $f\colon S^n \to S^n$ be a continuous function. If $\deg(f) \neq 0$ then f is surjective. Likewise, let $g\colon D^n \to D^n$ be a continuous function such that $g(S^{n-1}) \subseteq S^{n-1}$ and the resulting map $f = g_{|S^{n-1}}$ has non-trivial degree. Then g is surjective.*

Proof. Exercise 4 Sheet 11. □

4.7. **Theorem** (The fundamental theorem of algebra) [7] *Every non constant complex polynomial $P \in \mathbb{C}[X]$ has a root.*[8]

Proof. Let P be a polynomial of degree n with $n \geq 1$. Assume that P does not have a root. Then also P divided by its leading coefficient does not have a root, so we may assume that P is monic. The map f given by $x \mapsto \frac{P(x)}{\|P(x)\|}$ is a continuous map $\mathbb{C} \to S^1$ (here we have used that P does not have a root). Its restriction to S^1 is therefore null homotopic. However, we will show that the restricted map $f\colon S^1 \to S^1$ is also homotopic to $x \mapsto x^n$, which induces the multiplication by n map on $H_1(S^1) \cong \mathbb{Z}$, a contradiction. Let $P = \sum_{i=0}^n a_i X^i$ and consider the family of polynomials $P_t = \sum_{i=0}^n a_i t^{n-i} X^i$ with $t \in [0,1]$. We have $P_1 = P$ and $P_0 = X^n$ (recall that P is monic). Moreover, for $t > 0$, $P_t(z) = t^n P(\frac{z}{t})$ and hence has no zero. We may therefore consider $(x,t) \mapsto \frac{P_t(x)}{\|P_t(x)\|}$ as a continuous function on $S^1 \times [0,1] \to S^1$. It gives a continuous homotopy from f to $x \mapsto x^n$, showing that $f_{|S^1}$ has degree n as claimed. □

4.8. **Theorem** (The hairy ball theorem) *There exists a continuous function $s\colon S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ such that $\langle x, s(x)\rangle = 0$ if and only if n is odd. In particular, it does not exist for $n = 2$.*

In case $n = 2$, one imagines that this says that we cannot comb hair on a head (which we picture as S^2) in a continuous way as to lie nicely on our head. The set $\{(x,v) \in S^n \times \mathbb{R}^{n+1} \mid \langle x, v\rangle = 0\}$ of pairs of perpendicular vectors in $\mathbb{R}^{n+1}$, one of which is in S^n also has a name: it is called the tangent bundle TS^n of S^n. The hairy ball theorem therefore says that there is no section of the tangent bundle of an even dimensional sphere which does not attain the value 0 somewhere. Tangent bundles are topological spaces one can associate to smooth manifolds, and they allow for many more interesting invariants of smooth manifolds other than their homology groups. For instance, they often give rise to functionals on the homology groups.[9]

Proof. Suppose n is odd. For $x = (x_0, \ldots, x_n) \in S^n$ consider the element

$$s(x) = (-x_1, x_0, -x_3, x_2, \ldots, -x_n, x_{n-1}).$$

[7] Thanks to Panagiotis Papadopoulos for suggesting to improve the earlier proof.

[8] In other words, the field $\mathbb{C}$ is algebraically closed.

[9] These functionals are induced by what are called *characteristic classes*.

Then $\langle x, s(x)\rangle = 0$ and $s(x) \neq 0$ for all $x \in S^n$. Moreover, the so defined function s is continuous. Now assume that n is even and there exists a function $s\colon S^n \to \mathbb{R}^n \setminus \{0\}$ such that $\langle x, s(x)\rangle = 0$ for all $x \in S^n$. Consider the map

$$H\colon S^n \times [0,1] \to S^n, \quad (x,t) \mapsto \cos(\pi t)\cdot x + \sin(\pi t)\cdot \frac{s(x)}{\|s(x)\|}.$$

which is well-defined since x and $s(x)$ are perpendicular and $\sin(a)^2 + \cos(a)^2 = 1$ for any $a \in \mathbb{R}$. Then H is a homotopy from id_{S^n} to $-\mathrm{id}_{S^n}$. This contradicts that $\deg(\mathrm{id}_{S^n}) = 1 \neq (-1)^{n+1} = \deg(-\mathrm{id}_{S^n})$ since n is even. □

4.9. **Theorem** (The Borsuk–Ulam theorem) *Let $n \leq 2$. For every continuous map $f\colon S^n \to \mathbb{R}^n$ there exists $x \in S^n$ such that $f(x) = f(-x)$.*

For $n = 2$, this is often pictured as saying that there are always two antipodal points on the surface of the earth (i.e. two points that lie on the line through the centre of the earth) where the temperature and air pressure agree (one thinks that the temperature and the air pressure are continuous functions and one thinks that the surface of the earth is S^2).

Proof. Assume that for all $x \in S^n$, we have $f(x) \neq f(-x)$. Then the function

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

is a continuous function $S^n \to S^{n-1}$, equivariant with respect to the antipodal actions on both sides, i.e. $g(-x) = -g(x)$. For $n = 1$ this is a contradiction since S^1 is connected so the image of g must be a connected subset of S^0. For $n \geq 2$, by taking quotients with respect to the C_2-actions, we consider the induced map $\bar{g}\colon \mathbb{RP}^n \to \mathbb{RP}^{n-1}$. Consider the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(S^n) & \longrightarrow & \pi_1(\mathbb{RP}^n) & \xrightarrow{\mathfrak{m}_n} & C_2 & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle \bar{g}_*} & & \| & & \\
1 & \longrightarrow & \pi_1(S^{n-1}) & \longrightarrow & \pi_1(\mathbb{RP}^{n-1}) & \xrightarrow{\mathfrak{m}_{n-1}} & C_2 & \longrightarrow & 1
\end{array}$$

with rows as in Lemma 3.12. It is a direct check that the diagram commutes (it is clear for the left hand side, see Exercise 2 Sheet 9 for the right square). In particular, the middle vertical map is non-trivial (in fact an isomorphism if $n > 2$). When $n = 2$, the middle vertical map is isomorphic to $\mathbb{Z}/2 \to \mathbb{Z}$ which is the trivial map, a contradiction. □

4.10. **Remark** The Borsuk-Ulam theorem also holds for $n \geq 3$. Following the above proof, one can try show that there does not exist a continuous map $\mathbb{RP}^n \to \mathbb{RP}^{n-1}$ which induces an isomorphism on π_1. This will be shown next term making use of singular *cohomology* and its structure of a graded ring.

4.3. **Singular homology - Definitions and first examples.** Recall that homology ought to detect n-dimensional holes in the sense that it ought to have a particular value on spheres S^n. We recall that the sphere S^n is the boundary of the disk D^{n+1}. To define singular homology, it turns out to be useful to approximate spheres (and in fact all spaces) by simpler combinatorial spaces: In our case we wish to replace disks by simplices. Let us recall their definition.

4.11. **Definition** Let $n \geq 0$. The topological n-simplex Δ^n_{top} is the following topological space

$$\Delta^n_{\text{top}} = \{(x_0, \ldots, x_n) \in (\mathbb{R}_{\geq 0})^{n+1} \mid \sum_{i=0}^{n} x_i = 1\}.$$

4.12. **Example** We have $\Delta^0_{\text{Top}} = \{1\} \in \mathbb{R}$. Moreover, Δ^1_{Top} is homeomorphic to $[0,1]$ via the parametrization $t \mapsto (t, 1-t) \in \mathbb{R}^2$. Δ^2_{Top} looks like a filled triangle in $\mathbb{R}^3$, and Δ^3_{Top} like a solid tetrahedron. In general, Δ^n_{Top} is homeomorphic to D^n as well as to $[0,1]^n$.

4.13. **Definition** Let $n \geq 0$. Then the boundary $\partial\Delta^n_{\text{Top}}$ of the topological n-simplex is given by

$$\partial\Delta^n_{\text{Top}} = \{(x_0, \ldots, x_n) \in \Delta^n_{\text{Top}} \mid \exists i \in \{0, \ldots, n\} \text{ s.th. } x_i = 0\}.$$

4.14. **Example** We have $\partial\Delta^0_{\text{Top}} = \emptyset$, $\partial\Delta^1_{\text{Top}} \cong \{a, b\}$. In general $\partial\Delta^n_{\text{Top}}$ is the topological boundary in the technical sense, i.e. $\partial\Delta^n_{\text{Top}} = \Delta^n_{\text{Top}} \setminus \mathring{\Delta}^n_{\text{Top}}$. It is homeomorphic to S^{n-1}, as follows from Example 4.12 and the fact that a homeomorphism between topological spaces induces a homeomorphism of their respective boundaries.

It will turn out that it is convenient to think of simplices rather than disks. We record the following structure that the association $n \mapsto \Delta^n_{\text{Top}}$ has.

4.15. **Observation** Let $n \geq 0$ and let $0 \leq i \leq n$ and $0 \leq j \leq n-1$. Then there canonical maps

(1) $\delta_i \colon \Delta^{n-1}_{\text{Top}} \to \Delta^n_{\text{Top}}$, sending $(x_0, \ldots, x_{n-1})$ to the point $(x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_n)$, and
(2) $\sigma_i \colon \Delta^n_{\text{Top}} \to \Delta^{n-1}_{\text{Top}}$ sending $(x_0, \ldots, x_n)$ to $(x_0, \ldots, x_{i-1}, x_i + x_{i+1}, x_{i+2}, \ldots, x_n)$.

We note that $\partial\Delta^n_{\text{Top}} = \bigcup\limits_{0 \leq i \leq n} \delta_i(\Delta^{n-1}_{\text{Top}})$. Moreover, the above maps satisfy the following relations:

(1) $\delta_i\delta_j = \delta_{j+1}\delta_i$ as maps $\Delta^{n-1}_{\text{Top}} \to \Delta^{n+1}_{\text{Top}}$ whenever $i \leq j$,
(2) $\sigma_j\sigma_i = \sigma_i\sigma_{j+1}$ as maps $\Delta^{n+1}_{\text{Top}} \to \Delta^{n-1}_{\text{Top}}$ whenever $i \leq j$,
(3) $\sigma_j\delta_i = \begin{cases} \delta_i\sigma_{j-1} & \text{whenever } i < j \\ \mathrm{id}_{\Delta^n_{\text{Top}}} & \text{whenever } i = j, j+1 \\ \delta_{i-1}\sigma_j & \text{whenever } j+1 < i \end{cases}$

It will also be convenient to gather the above data into compact categorical language. To this end, we define a category Δ, the *simplex category*, to have objects $[n]$ for $n \geq 0$ and whose morphisms are generated by maps $\delta_i \colon [n-1] \to [n]$ and $\sigma_j \colon [n] \to [n-1]$ satisfying the above relations. With this, we obtain a functor $\Delta_{\text{Top}} \colon \Delta \to \text{Top}$, sending $[n]$ to Δ^n_{Top}.

4.16. **Lemma** *The category Δ is concretely given by the category whose objects are the non-empty finite linearly ordered sets $[n] = \{0, \ldots, n\}$ and whose morphisms $f \colon [n] \to [m]$ are monotone maps, that is, $f(a) \leq f(a')$ whenever $a \leq a'$.*

Proof. Any monotone map $f \colon [n] \to [m]$ is uniquely given by a surjection followed by an injection. Any surjection is a composite of σ_j's and any injection is a composite of δ_i's, where $\sigma_j \colon [n] \to [n-1]$ is the unique surjective map with $\sigma_j(j) = \sigma_j(j+1) = j$ and δ_i is the unique injective map $[n-1] \to [n]$ such that $i \notin \delta_i([n-1])$. The so constructed maps δ_i and σ_j satisfy the above observed relations. □

4.17. **Notation** Let $\mathcal{C}$ be a category. An object of $\mathrm{Fun}(\Delta, \mathcal{C})$ is called a *cosimplicial object* in $\mathcal{C}$, and an object of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ is called a *simplicial object* in $\mathcal{C}$. In particular, the collection of the topological n-simplices Δ^n_{Top} form a cosimplicial topological space, i.e. a cosimplicial object in the category of topological spaces.

The idea now is to use simplices to measure to what extent a topological space admits holes of dimension n. Since singular homology ought to take values in abelian groups, we simply consider the following abelian group:

4.18. **Definition** Let $n \geq 0$. We define the abelian group $C^{\mathrm{sing}}_n(X)$ of *singular n-simplices in X* to be

$$C^{\mathrm{sing}}_n(X) = \mathbb{Z}[\mathrm{Hom}_{\mathrm{Top}}(\Delta^n_{\mathrm{Top}}, X)],$$

i.e. the free abelian group generated by all continuous maps $\sigma\colon \Delta^n_{\mathrm{Top}} \to X$. For $n < 0$, we set $C^{\mathrm{sing}}_n(X) = 0$.

An element in $C^{\mathrm{sing}}_n(X)$ is therefore a finite linear combination (with integer coefficients) of maps $\sigma\colon \Delta^n_{\mathrm{Top}} \to X$. Since we wish to detect n-dimensional holes and since $\partial\Delta^{n+1}_{\mathrm{Top}}$ ought to have such an n-dimensional hole, we will not be too interested in all singular n-simplices, but rather only those, which look like $\partial\Delta^n_{\mathrm{Top}}$ is the following way. We note that for $0 \leq i \leq n$, the maps $\delta_i\colon \Delta^{n-1}_{\mathrm{Top}} \to \Delta^n_{\mathrm{Top}}$ induce maps $\delta_i^*\colon C^{\mathrm{sing}}_n(X) \to C^{\mathrm{sing}}_{n-1}(X)$ sending σ to $\sigma \circ \delta_i$.

4.19. **Definition** For $n \geq 0$, we define a map $d_n\colon C^{\mathrm{sing}}_n(X) \to C^{\mathrm{sing}}_{n-1}(X)$ as $d_n = \sum_{i=0}^n (-1)^i \delta_i^*$. A singular n-simplex $x \in C^{\mathrm{sing}}_n(X)$ is called *closed* if $d_n(x) = 0$. The collection of closed singular n-simplices $Z_n(X)$ forms an abelian subgroup of $C^{\mathrm{sing}}_n(X)$ called the abelian group of singular n-cycles.

4.20. **Example** Since $C_{-1}(X) = 0$, we may set $d_0\colon C^{\mathrm{sing}}_0(X) \to C^{\mathrm{sing}}_{-1}(X)$ to be the zero map. The map $d_1\colon C^{\mathrm{sing}}_1(X) \to C^{\mathrm{sing}}_0(X)$ sends $\sigma\colon \Delta^1_{\mathrm{Top}} \cong [0,1] \to X$ to $\sigma(0) - \sigma(1)$. Hence d_1 sees whether or not a path in X is closed. Similarly, $d_2\colon C^{\mathrm{sing}}_2(X) \to C^{\mathrm{sing}}_1(X)$ takes $\sigma\colon \Delta^2_{\mathrm{Top}} \to X$ to the $\sigma_{12} - \sigma_{02} + \sigma_{01}$, where the subscripts refer to the restriction of σ to the interval between the two indicated edges of Δ^2_{Top}.

We then note that a closed singular n-simplex is an abstract way to see a potential n-dimensional hole in X. But we do not want to measure potential n-dimensional holes in X, but rather actual such holes. To do so, we wish to discard closed singular n-simplices if they can be filled in the following sense.

4.21. **Definition** For $n \geq 0$, we say that the image $B_n(X)$ of $d_{n+1}\colon C^{\mathrm{sing}}_{n+1}(X) \to C^{\mathrm{sing}}_n(X)$ consists of *boundary* singular n-simplices.

4.22. **Lemma** *For any $n \geq 0$, the composite*

$$C^{\mathrm{sing}}_{n+1}(X) \xrightarrow{d_{n+1}} C^{\mathrm{sing}}_n(X) \xrightarrow{d_n} C^{\mathrm{sing}}_{n-1}(X)$$

is the zero map.

Proof. It suffices to show that this is true on generators of the free abelian group $C^{\mathrm{sing}}_{n+1}(X)$, which are given by $\sigma\colon \Delta^n_{\mathrm{Top}} \to X$. By definition, we have

$$d_n(d_{n+1}(\sigma)) = d_n(\sum_{i=0}^{n+1}(-1)^i \sigma \circ \delta_i) = \sum_{j=0}^{n}\sum_{i=0}^{n+1}(-1)^{i+j}\sigma \circ \delta_i \circ \delta_j.$$

Now, we recall the relations among the δ_i's observed in Observation 4.15. We then have

$$\begin{aligned}
d_n(d_{n+1}(\sigma)) &= \sum_{j=0}^{n}\sum_{i=0}^{n+1}(-1)^{i+j}\sigma \circ \delta_i \circ \delta_j \\
&= \sum_{0\leq i\leq j\leq n}(-1)^{i+j}\sigma\delta_i\delta_j + \sum_{0\leq j<i\leq n+1}(-1)^{i+j}\sigma\delta_i\delta_j \\
&= \sum_{0\leq i\leq j\leq n}(-1)^{i+j}\sigma\delta_{j+1}\delta_i + \sum_{0\leq j<i\leq n+1}(-1)^{i+j}\sigma\delta_i\delta_j \\
&= \sum_{0\leq i<j\leq n+1}(-1)^{i+j+1}\sigma\delta_j\delta_i + \sum_{0\leq j<i\leq n+1}(-1)^{i+j}\sigma\delta_i\delta_j \\
&= 0
\end{aligned}$$

as needed. □

With this at hand we define the singular homology of a space X as follows.

4.23. **Definition** Let X be a topological space and $n \in \mathbb{Z}$. Define $H_n(X) = Z_n(X)/B_n(X)$ as the quotient abelian group of the inclusion $B_n(X) \subseteq Z_n(X)$.

We gather the following immediate consequences of the definition.

4.24. **Remark** (1) The association $X \mapsto H_n(X)$ is a functor $\mathrm{Top} \to \mathrm{Ab}$.

(2) $H_n(\emptyset) = 0$ for all $n \in \mathbb{Z}$.
(3) For $n < 0$, we have $H_n(X) = 0$.
(4) For $n = 0$, we have $H_0(X) \cong \mathbb{Z}[\pi_0(X)]$. Indeed, $Z_0(X) = C^{\mathrm{sing}}_0(X) = \mathbb{Z}[X]$ is the free abelian group on the set X and $B_0(X)$ consists of the elements of the form $x - y$ whenever there is a path between x and y. Therefore $H_0(X)$ is the quotient of $\mathbb{Z}[X]$ by the equivalence $x \sim y$ whenever $[x] = [y] \in \pi_0(X)$. Since $\mathbb{Z}[-]\colon \mathrm{Set} \to \mathrm{Ab}$ is a left adjoint, it preserves quotients by equivalence relations, and we obtain $H_0(X) \cong \mathbb{Z}[\pi_0(X)]$ as claimed.
(5) $H_n(*) = 0$ for $n \neq 0$. Indeed, we have $C^{\mathrm{sing}}_n(*) \cong \mathbb{Z}$ for all $n \geq 0$. Moreover, the map $d_n\colon C^{\mathrm{sing}}_n(*) \to C^{\mathrm{sing}}_n(*)$ is given $\sum_{i=0}^n(-1)^n d_i$ where each d_i is the identity map. In particular, the map d_n is an isomorphism for $n > 0$ even and the zero map for $n > 0$ odd. Therefore $Z_n(*) = 0$ if $n > 0$ is even and $B_n(*) = Z_n(*)$ if $n > 0$ is odd. Consequently, $H_n(*) = 0$ for all $n > 0$.

Indeed, the decisive part for (1) is that the maps $d_n\colon C_n^{\mathrm{sing}}(X) \to C_{n-1}^{\mathrm{sing}}(X)$ are natural in X. That is, given $f\colon X \to Y$, the square

$$\begin{array}{ccc} C_n^{\mathrm{sing}}(X) & \longrightarrow & C_{n-1}^{\mathrm{sing}}(X) \\ \downarrow & & \downarrow \\ C_n^{\mathrm{sing}}(Y) & \longrightarrow & C_{n-1}^{\mathrm{sing}}(Y) \end{array}$$

commutes. This follows essentially from the fact that for $\sigma\colon \Delta^n_{\mathrm{Top}} \to X$ and $0 \leq i \leq n$, we have $(f \circ \sigma) \circ \delta_i = f \circ (\sigma \circ \delta_i)$.

4.25. **Proposition** *Let X be a path connected topological space and $x \in X$. There is a canonical group homomorphism $h\colon \pi_1(X,x) \to H_1(X)$. This map induces an isomorphism $\pi_1(X,x)^{\mathrm{ab}} \cong H_1(X)$.*

Proof. Recall that $\pi_1(X,x)$ is the set of homotopy rel endpoint classes of paths $\gamma\colon [0,1] \to X$ with $\gamma(0) = x = \gamma(1)$. Recall that $[0,1] \cong \Delta^1_{\mathrm{Top}}$. Under this homeomorphism, we can send γ to $\gamma \in C_1^{\mathrm{sing}}(X)$ and observe that it lies in $Z_1(X)$ since $d_1(\gamma) = \gamma(0) - \gamma(1) = 0$. We wish to show that its class in $H_1(X)$ is independent of the choice of representative γ of $[\gamma] \in \pi_1(X,x)$ and that the resulting association is a group homomorphism. First, we note that for γ and γ' composable paths, there is a continuous map $\sigma\colon \Delta^2_{\mathrm{Top}} \to X$ such that $d_2(\sigma) = \gamma'$, $d_1(\sigma) = \gamma \star \gamma'$ and $d_0(\sigma) = \gamma$. We deduce that $\gamma - \gamma' \star \gamma + \gamma' = 0$ in $H_1(X)$ so that $[\gamma] + [\gamma'] = [\gamma' \star \gamma] \in H_1(X)$. This applies in particular to closed paths γ and γ' at $x \in X$. In particular, we deduce that $[\mathrm{const}_x] = 0 \in H_1(X)$ since $\mathrm{const}_x \star \mathrm{const}_x = \mathrm{const}_x$. Moreover, suppose $[\gamma] = [\gamma'] \in \pi_1(X,x)$. Pick a homotopy rel endpoints $H\colon [0,1] \times [0,1] \to X$ between γ and γ'. There is then a quotient map $[0,1] \times [0,1] \to \Delta^2_{\mathrm{Top}}$ which collapses the subspace $\{1\} \times [0,1]$. Since H is a pointed homotopy, it factors through this quotient map and the resulting map $\sigma\colon \Delta^2_{\mathrm{Top}} \to X$ has the property that its boundary 1-simplices are given by γ, γ' and const_x. We deduce that $\gamma - \gamma' + \mathrm{const}_x = 0 \in H_1(X)$ and therefore that $\gamma = \gamma'$ in $H_1(X)$. In total, this shows that there is a well-defined map $\pi_1(X,x) \to H_1(X)$ which sends a $\pi_1(X,x)$ representative γ to the image of γ in $H_1(X)$, and that this map is a group homomorphism. Since $H_1(X)$ is abelian, we obtain a canonical induced map $\pi_1(X,x)^{\mathrm{ab}} \to H_1(X)$. We now show that this map is an isomorphism. To do so, we simply construct an inverse as follows. For any point $y \in X$, we fix a path α_y from x to y, and we choose $\alpha_x = \mathrm{const}_x$. For a singular 1-simplex $f\colon \Delta^1_{\mathrm{Top}} \to X$, we may consider the loop $\alpha^{-1}_{f(1)} \star f \star \alpha_{f(0)}$. Since $C_1^{\mathrm{sing}}(X)$ is a free abelian group on 1-simplices, these choices give a group homomorphism $\varphi\colon C_1^{\mathrm{sing}}(X) \to \pi_1(X,x)^{\mathrm{ab}}$. We now claim that the composite

$$C_2^{\mathrm{sing}}(X) \xrightarrow{d_2} C_1^{\mathrm{sing}}(X) \xrightarrow{\varphi} \pi_1(X,x)^{\mathrm{ab}}$$

is the trivial homomorphism. To see this, it suffices to show it on generators of the source. So consider $\sigma\colon \Delta^2_{\mathrm{Top}} \to X$ and let $\sigma_i = \sigma \circ \delta_i$). Then we get

$$\begin{aligned}\varphi(d_2(\sigma)) &= \varphi(\sigma_0 + \sigma_2 - \sigma_1)\\ &= (\alpha^{-1}_{\sigma_0(1)} \star \sigma_0 \star \alpha_{\sigma_0(0)}) \star (\alpha^{-1}_{\sigma_2(1)} \star \sigma_2 \star \alpha_{\sigma_2(0)}) \star (\alpha^{-1}_{\sigma_1(0)} \star \sigma_1^{-1} \star \alpha_{\sigma_1(1)})\\ &= \alpha^{-1}_{\sigma_0(1)} \star \sigma_0 \star \sigma_2 \star \sigma_1^{-1} \star \alpha_{\sigma_0(1)}\\ &= \mathrm{const}_x\end{aligned}$$

in $\pi_1(X,x)^{\mathrm{ab}}$. Here, the third equality holds since $\sigma_0(0) = \sigma_2(1)$, $\sigma_2(0) = \sigma_1(0)$, and $\sigma_0(1) = \sigma_1(1)$, and the last equality holds since σ witnesses that the two paths $\sigma_0 \star \sigma_2$ and σ_1 are homotopic rel endpoints. Consequently, φ descends to a well-defined map $H_1(X) \to \pi_1(X,x)^{\mathrm{ab}}$. The composite $\pi_1(X,x)^{\mathrm{ab}} \to H_1(X) \to \pi_1(X,x)^{\mathrm{ab}}$ is the identity by construction. It therefore suffices to show that $\pi_1(X,x) \to H_1(X)$ is surjective. So let $t \in Z_n(X)$ be an arbitrary element, written as $\sum_i n_i f_i$ with $f_i\colon \Delta^1_{\mathrm{Top}} \to X$. That $d_1(t) = 0$ means that the image of the f_i's give rise to several closed loops in X. For each of such closed loops, we can choose some i and a path from x to $f_i(0)$. Conjugating the loop by this path doesn't change the class in $H_1(X)$ and shows that each such path is in the image of the map h, and since h is a group homomorphism, we finally deduce that h is surjective and hence an isomorphism. □

4.26. **Remark** The homomorphism $h\colon \pi_1(X,x) \to H_1(X)$ is called the *Hurewicz* homomorphism. It exists more generally for all $n \geq 1$, i.e. there are group homomorphisms $h_n\colon \pi_n(X,x) \to H_n(X)$. The above proposition is then part of the *Hurewicz theorem* which we will prove later. In addition to Proposition 4.25 it states the following. Suppose X is a simply connected[10] topological space and $n \geq 2$. If $\pi_i(X,x) = 0$ for all $1 \leq i \leq n-1$, then $H_i(X) = 0$ for all $1 \leq i \leq n-1$. Moreover, in this case, the Hurewicz homomorphism $h_n\colon \pi_n(X,x) \to H_n(X)$ is an isomorphism and the Hurewicz homomorphism $h_{n+1}\colon \pi_{n+1}(X,x) \to H_{n+1}(X)$ is surjective.

In general, the Hurewicz homomorphism $\pi_2(X,x) \to H_2(X)$ is not surjective, even if $\pi_1(X,x) \to H_1(X)$ is an isomorphism. For instance, this is the case for $X = T^2$ the 2-torus: We have seen already that $\pi_2(T^2) = 0$, but we will show later that $H_2(T^2) \cong \mathbb{Z}$.

4.27. **Corollary** *We have that* $H_1(S^1) \cong \mathbb{Z}$, $H_1(S^n) = 0$ *for* $n \geq 2$, $H_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$ *for* $n \geq 2$, $H_1(\mathbb{CP}^n) = 0$ *and* $H_1(\mathbb{HP}^n) = 0$ *for* $n \geq 1$, $H_1(T^2) = \mathbb{Z}^2$, $H_1(K) = (\mathbb{Z} \rtimes \mathbb{Z})^{\mathrm{ab}} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $H_1(\Sigma_g) = \mathbb{Z}^{2g}$, $H_1(\bigvee_S S^1) = \mathbb{Z}^{|S|}$.

Proof. This is an exercise in calculating abelianizations of groups with explicit presentations. The only group where we have not discussed a presentation is $\mathbb{Z} \rtimes \mathbb{Z}$ which has the following presentation $\mathbb{Z} \rtimes \mathbb{Z} \cong \langle a, b \mid abab^{-1} \rangle$. This shows that $(\mathbb{Z} \rtimes \mathbb{Z})^{\mathrm{ab}}$ has a presentation $\langle a, b \mid aba^{-1}b^{-1}, 2a \rangle$ showing the claim. □

4.28. **Corollary** *Let X be a path connected space. Then $H_1(\Sigma(X)) = 0$. For X and Y path connected CW complexes, we have $H_1(X \vee Y) = H_1(X) \oplus H_1(Y)$. For X and Y path connected spaces, we have $H_1(X \times Y) = H_1(X) \times H_1(Y)$.*

Proof. The first follows from $\pi_1(\Sigma(X), x) = 0$, see Example 2.67. For the second, recall from Example 2.68 that $\pi_1(X \vee Y) \cong \pi_1(X) \star \pi_1(Y)$. The functor $(-)^{\mathrm{ab}}\colon \mathrm{Grp} \to \mathrm{Ab}$ is left adjoint to the inclusion and hence preserves colimits, in particular coproducts. Now use that in Ab

[10]Recall that this means that X is path connected and $\pi_1(X,x) = 1$ for some $x \in X$.

coproducts are given by the direct sum of abelian groups. For the final claim, recall from Lemma 2.24 (2) that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$. Then we can use that $(-)^{\mathrm{ab}} \colon \mathrm{Grp} \to \mathrm{Ab}$ also commutes with finite products (Exercise), and that finite products in Ab are also given by the direct sum. □

The following is essentially the only computation we can do using the definition of singular homology.

4.29. **Proposition** *Let $X \subseteq \mathbb{R}^n$ be a star-shaped subset with $x \in X$ a star. Then $H_n(X) = 0$ for $n > 0$.*

Proof. Being star-shaped with star x means that for all $y \in X$, the straight line between x and y (inside $\mathbb{R}^n$) is contained in X. Note that Δ^{n+1} consists of all straight lines between Δ^n and $(1, 0, \dots, 0) \in \mathbb{R}^{n+2}$. Hence, given an n-simplex $\sigma \colon \Delta^n_{\mathrm{Top}} \to X$, we obtain a map $h_n(\sigma) \colon \Delta^{n+1}_{\mathrm{Top}} \to X$ by requiring that its restriction along δ_0 to Δ^n_{Top} is σ, that the opposite vertex $(1, 0 \dots, 0)$ is sent to x, and straight lines between two points in $\Delta^{n+1}_{\mathrm{Top}}$ are sent to straight lines in X. Sending σ to $h_n(\sigma)$ defines homomorphisms $h_n \colon C^{\mathrm{sing}}_n(X) \to C^{\mathrm{sing}}_{n+1}(X)$ which for $n > 0$ satisfy the relation

$$d_{n+1}h_n + h_{n-1}d_n = \mathrm{id}$$

as one readily calculates. Hence, if $\sigma \in Z_n(X)$ and $n > 0$, we find

$$\sigma = d_{n+1}h_n(\sigma) + h_{n-1}d_n(\sigma) = d_{n+1}h_n(\sigma) \in B_n(X)$$

so that $H_n(X) = 0$ as claimed. □

4.30. **Example** Typical examples of star-shaped sets are the spaces Δ^n_{Top} for $n \geq 0$ or other convex subsets of $\mathbb{R}^n$. The product of any two star-shaped subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ is canonically a star-shaped subset in $\mathbb{R}^{n+m}$. In particular, $\Delta^n_{\mathrm{Top}} \times \Delta^m_{\mathrm{Top}}$ is star-shaped.

In order to formulate all properties that singular homology enjoys, it is beneficial to introduce a relative version of singular homology.

4.31. **Definition** Let $A \subseteq X$ be a subspace of a topological space X. We define the relative singular n-simplices $C^{\mathrm{sing}}_n(X, A)$ of the pair (X, A) to be the quotient abelian group $C^{\mathrm{sing}}_n(X)/C^{\mathrm{sing}}_n(A)$.

4.32. **Lemma** *The abelian group $C^{\mathrm{sing}}_n(X, A)$ is free abelian. There is a unique map $d_n \colon C^{\mathrm{sing}}_n(X, A) \to C^{\mathrm{sing}}_{n-1}(X, A)$ making the diagram*

$$\begin{array}{ccc} C^{\mathrm{sing}}_n(X) & \xrightarrow{d_n} & C^{\mathrm{sing}}_{n-1}(X) \\ \downarrow & & \downarrow \\ C^{\mathrm{sing}}_n(X, A) & \xrightarrow{d_n} & C^{\mathrm{sing}}_{n-1}(X, A) \end{array}$$

commute. It again satisfies $d_n \circ d_{n+1} = 0$.

Proof. It is not hard to check that $C^{\mathrm{sing}}_n(X, A)$ is the free abelian group on those simplices $\sigma \colon \Delta^n_{\mathrm{Top}} \to X$ whose image is not contained in A, simply because this is the complement of all n-simplices whose image does lie in A. Now we consider the commutative diagram

$$\begin{array}{ccccc} C_n^{\mathrm{sing}}(A) & \longrightarrow & C_n^{\mathrm{sing}}(X) & \longrightarrow & C_n^{\mathrm{sing}}(X,A) \\ \downarrow & & \downarrow & & \vdots \\ C_{n-1}^{\mathrm{sing}}(A) & \longrightarrow & C_{n-1}^{\mathrm{sing}}(X) & \longrightarrow & C_{n-1}^{\mathrm{sing}}(X,A) \end{array}$$

in which the left square commutes by naturality of the map d_n. Hence, there exists a dashed arrow making the right square commute. In addition, it is uniquely determined by the commutativity since the upper horizontal map is surjective. To see that it again satisfies $d_n \circ d_{n+1} = 0$, we again use this surjectivity to deduce from the case of X. □

4.33. **Definition** Let (X, A) be a pair of spaces and $n \geq 0$. We define the relative singular homology $H_n(X, A)$ to be the quotient $\ker(d_n)/\mathrm{Im}(d_{n+1})$.

Note that $H_n(X, \emptyset) = H_n(X)$ by definition since $C_n^{\mathrm{sing}}(\emptyset) = 0$ for all $n \geq 0$. We are now ready to state all properties singular homology enjoys.

4.4. **Singular homology - Properties.**

4.34. **Theorem** *The association $(X, A) \mapsto \{H_n(X, A)\}_{n\in\mathbb{Z}}$ satisfies the following properties.*

(1) *$H_n(\emptyset) = 0$ for all $n \in \mathbb{Z}$ and $H_n(*) = 0$ for all $0 \neq n \in \mathbb{Z}$ and $H_0(*) \cong \mathbb{Z}$.*
(2) *If f and g are homotopic maps of pairs $(X, A) \to (Y, B)$, then they induce the same map $H_n(X, A) \to H_n(Y, B)$.*
(3) *For any family of topological spaces $\{X_i\}_{i\in I}$ and all $n \in \mathbb{Z}$, the evident map*

$$\bigoplus_{i\in I} H_n(X_i) \to H_n(\coprod_{i\in I} X_i)$$

is an isomorphism.
(4) *For any $n \in \mathbb{Z}$ and pair (X, A) there exists a boundary map $\partial_n \colon H_n(X, A) \to H_{n-1}(A)$ making the following sequence* exact*.*

$$H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{p_*} H_n(X, A) \xrightarrow{\partial_n} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X)$$

(5) *For a pair (X, A) and $U \subseteq A$ such that $\overline{U} \subseteq \mathring{A}$ and all $n \in \mathbb{Z}$, the evident inclusion induces an isomorphism*

$$H_n(X \setminus U, A \setminus U) \xrightarrow{\cong} H_n(X, A).$$

(6) *For a space X and subspaces $A, B \subseteq X$ such that $\mathring{A} \cup \mathring{B} = X$ and any $n \in \mathbb{Z}$, there is a natural*[11] *boundary map $\partial_n \colon H_n(X) \to H_{n-1}(A \cap B)$ making the following sequence exact.*

$$H_n(A\cap B) \xrightarrow{(\iota_*^A, \iota_*^B)} H_n(A) \oplus H_n(B) \xrightarrow{(j_*^A - j_*^B)} H_n(X) \xrightarrow{\partial_n} H_{n-1}(A\cap B) \to H_{n-1}(A) \oplus H_{n-1}(B)$$

(7) *For non-empty X, there is a natural suspension isomorphism $H_{n+1}(\Sigma(X), N) \cong H_n(X, x)$.*

[11] see Warning 4.59 for a discussion of the naturality that ∂ satisfies.

4.35. **Corollary** *Let (X, x) and (Y, y) be pointed spaces such that the respective basepoints have pointed contractible open neighborhoods*[12] *and $n \geq 0$. Then there is a canonical isomorphism $\widetilde{H}_n(X) \oplus \widetilde{H}_n(Y) \cong \widetilde{H}_n(X \vee Y)$.*

Proof. Exercise. □

We will now move towards proving the above properties, then continue with calculations of various homology groups as well as a description of singular homology for CW complexes called cellular homology. To address the first properties, it will be convenient to discuss the notions of chain complexes and general statements therein. We formulate everything for modules over a fixed base ring, the main case of interest being the case of the integers.

4.36. **Definition** Let R be a ring. A chain complex $(M_\bullet, d_\bullet)$ of R-modules[13] consists of a sequence of R-modules $\{M_n\}_{n\in\mathbb{Z}}$ together with maps $d_n : M_n \to M_{n-1}$ such that $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$. We often write

$$\cdots \to M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \to \ldots$$

for such a chain complex.

We define the homology of $(M_\bullet, d_\bullet)$ to be the family of R-modules given by $\{H_n(M_\bullet, d_\bullet) = \ker(d_n)/\mathrm{Im}(d_{n+1})\}_{n\in\mathbb{Z}}$. We say that a chain complex is *exact at place n* if $H_n(M_\bullet, d_\bullet) = 0$, and simply *exact* if it is exact at every $n \in \mathbb{Z}$. An exact chain complex is also called a *long exact sequence*.

4.37. **Remark** An exact chain complex of the form

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$$

is called a *short exact sequence*. Concretely, this means that i is injective, p is surjective and $\ker(p) = \mathrm{Im}(i)$. An even shorter exact sequence, i.e. an exact chain complex of the form

$$0 \to M \xrightarrow{f} M' \to 0$$

is one where f is an isomorphism. An extremely short exact sequence, i.e. one of the form

$$0 \to M \to 0$$

is one where $M = 0$.

4.38. **Example** (1) Let X be a topological space. Then $(C_\bullet^{\mathrm{sing}}, d_\bullet)$ is a chain complex of abelian groups (i.e. $\mathbb{Z}$-modules). The homology of this chain complex is precisely the singular homology $H_\bullet(X)$ of X.

(2) Likewise, for a pair (X, A), we have that $(C_\bullet^{\mathrm{sing}}(X, A), d_\bullet)$ is a chain complex of abelian groups. Its homology is the relative singular homology $H_\bullet(X, A)$ of (X, A).

As always, we shall be interested not only in chain complexes, but also in morphisms of chain complexes.

[12]E.g. CW complexes.

[13]We work with left R-modules.

4.39. **Definition** Let $(M_\bullet, d_\bullet)$ and $(M'_\bullet, d'_\bullet)$ be chain complexes of R-modules. A chain map $f_\bullet$ consists of R-linear maps $f_n \colon M_n \to M'_n$ for all $n \in \mathbb{Z}$ such that for all $n \in \mathbb{Z}$, the squares

$$\begin{array}{ccc} M_n & \xrightarrow{d_n} & M_{n-1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ M'_n & \xrightarrow{d'_n} & M'_{n-1} \end{array}$$

commute. We write $\mathrm{Ch}(R)$ for the category of chain complexes of R-modules and chain maps. An isomorphism in $\mathrm{Ch}(R)$ is equivalently a chain map $f_\bullet$ such that f_n is an isomorphism of R-modules for all $n \in \mathbb{Z}$.

4.40. **Lemma** *For every $n \in \mathbb{Z}$, the association $(M_\bullet, d_\bullet) \mapsto H_n(M_\bullet, d_\bullet)$ refines to a functor $H_n(-) \colon \mathrm{Ch}(R) \to \mathrm{Mod}(R)$.*

Proof. By definition, a chain map $f_\bullet \colon (M_\bullet, d_\bullet) \to (M'_\bullet, d'_\bullet)$ induces maps $\ker(d_n) \to \ker(d'_n)$ and $\mathrm{Im}(d_{n+1}) \to \mathrm{Im}(d'_{n+1})$. Consequently, it also induces a map $H_n(f) \colon H_n(M_\bullet) \to H_n(M'_\bullet)$. It then follows readily from the definitions that $H_n(\mathrm{id}) = \mathrm{id}$ and $H_n(g \circ f) = H_n(g) \circ H_n(f)$. □

4.41. **Definition** Let $f_\bullet \colon M_\bullet \to M'_\bullet$[14] be a chain map. Then f is called a *quasi-isomorphism* if it induces an isomorphism on all homology groups, i.e. if $H_n(f)$ is an isomorphism for all $n \in \mathbb{Z}$.

4.42. **Example** Any isomorphism is a quasi-isomorphism, but the converse is not true: For instance, the unique map from the chain complex

$$\cdots \to 0 \to M \xrightarrow{\mathrm{id}} M \to 0 \to \dots$$

to the 0-chain complex is a quasi-isomorphism for any R-module M.

4.43. **Definition** Let $f, g \colon M \to M'$ be two chain maps. A chain homotopy between f and g consists of maps $h_n \colon M_n \to M'_{n+1}$ for all $n \in \mathbb{Z}$ satisfying the relation

$$d_{n+1} h_n + h_{n-1} d_n = f_n - g_n.$$

A chain homotopy equivalence is a map $f \colon M \to M'$ such that there exists a map $g \colon M' \to M$ such that fg and gf are chain homotopic to $\mathrm{id}_{M'}$ and id_M, respectively.

4.44. **Lemma** *Chain homotopic maps induce the same map on homology. In particular, a chain homotopy equivalence is a quasi-isomorphism.*

Proof. Let $x \in \ker(d_n)$ represent an element of $H_n(M)$. Then we have

$$f_n(x) = g_n(x) + d_{n+1} h_n(x) + h_{n-1} d_n(x) = g_n(x) + d_{n+1} h_n(x)$$

so indeed $[f_n(x)] = [g_n(x)] \in H_n(M')$. The second claim then follows immediately. □

4.45. **Example** There are chain maps which are quasi-isomorphisms but not chain homotopy equivalences: For instance, the chain complex

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to 0 \to \dots$$

[14] As usual, we start being slightly sloppy in notation and don't always add the differentials explicitly.

with non-trivial terms in degree 1 and 0 is quasi-isomorphic to the chain complex concentrated in degree 0 and given by $\mathbb{Z}/p\mathbb{Z}$ there. However, every chain map in the other direction is the zero map. This implies that the two complexes are not chain homotopy equivalent.

4.46. **Example** Let $X \subseteq \mathbb{R}^n$ be a starshaped subset with star $x \in X$. Then the canonical map $C_\bullet^{\text{sing}}(X) \to \mathbb{Z}$, where we view the latter as a complex concentrated in degree 0, induced by the map $C_0^{\text{sing}}(X) \to H_0(X) \cong \mathbb{Z}$ is a chain homotopy equivalence. Indeed, consider the map $\mathbb{Z} \to C_\bullet^{\text{sing}}(X)$ given by sending 1 to the 0-simplex $\{x\} \subseteq X$. Then the composite $\mathbb{Z} \to C_\bullet^{\text{sing}}(X) \to \mathbb{Z}$ is the identity. We claim that the maps $h_n \colon C_n^{\text{sing}}(X) \to C_{n+1}^{\text{sing}}(X)$ from Proposition 4.29 form a chain homotopy between the identity and the composite $C_\bullet^{\text{sing}}(X) \to \mathbb{Z} \to C_\bullet^{\text{sing}}(X)$. Indeed, we have verified in Proposition 4.29 that for $n > 0$, we have

$$d_{n+1}h_n + h_{n-1}d_n = \text{id} = \text{id} - 0$$

and d_1h_0 is the map sending a point y to the difference $y - x$ which is the difference between the identity of $C_0^{\text{sing}}(X)$ and the above described composite $C_0^{\text{sing}}(X) \to \mathbb{Z} \to C_0^{\text{sing}}(X)$ as needed.

After this small digression on chain complexes, we come back to singular homology:

4.47. **Lemma** *The association* $(X, A) \mapsto C_\bullet^{\text{sing}}(X, A)$ *refines to a functor from pairs of topological spaces to* $\text{Ch}(\mathbb{Z})$*.*

Proof. The argument from Remark 4.24 shows that $X \mapsto C_\bullet^{\text{sing}}(X)$ refines to a functor Top $\to$ $\text{Ch}(\mathbb{Z})$. It follows that a map of pairs $(X, A) \to (Y, B)$ induces a commutative square

$$\begin{array}{ccc} C_\bullet^{\text{sing}}(A) & \longrightarrow & C_\bullet^{\text{sing}}(X) \\ \downarrow & & \downarrow \\ C_\bullet^{\text{sing}}(B) & \longrightarrow & C_\bullet^{\text{sing}}(Y) \end{array}$$

and hence induces a canonical map $C_\bullet^{\text{sing}}(X, A) \to C_\bullet^{\text{sing}}(Y, B)$ which one readily checks to be functorial. □

Let us now introduce homology with coefficients in an arbitrary abelian group M.

4.48. **Definition** Let (X, A) be a pair of spaces and M an abelian group. We define $C_\bullet^{\text{sing}}(X, A; M) = C_\bullet^{\text{sing}}(X, A) \otimes_{\mathbb{Z}} M$ and $H_\bullet(X, A; M)$ as its homology.

4.49. **Remark** Note that for any chain complex $M_\bullet$, the tensor product with any abelian group is indeed again a chain complex. Moreover, if M is an R-module for some ring R, then $C_\bullet^{\text{sing}}(X, A; M)$ is a chain complex of R-modules and its homology groups are therefore also R-modules. In particular, if K is a field like $\mathbb{F}_p$ or $\mathbb{Q}$, then $H_\bullet(X, A; K)$ consists of K-vector spaces. Moreover, the formation of the singular chain complex with coefficients in M gives rise to a functor Pairs $\to \text{Ch}(R)$ whenever M is an R-module.

4.50. **Remark** In fact, for a commutative ring R like the integers $\mathbb{Z}$, there is a canonical symmetric monoidal structure on $\text{Ch}(R)$ given as follows: For chain complexes $C_\bullet$ and $D_\bullet$ one defines a *double complex* (i.e. an chain complex of chain complexes) with (p, q) entry given by $C_p \otimes_R D_q$ and horizontal and vertical maps given by the respective differentials. For a double complex one can construct its *total complex* which sums up terms of fixed total degree: Concretely, we obtain

$$(C \otimes_R D)_n = \bigoplus_{p+q=n} C_p \otimes_R D_q.$$

The differential $\partial^\otimes$ on $C \otimes_R D$ is determined by its restriction to $C_p \otimes_R D_q$ and on elementary tensors is given by

$$\partial_n^\otimes(x \otimes y) = d_n(x) \otimes y + (-1)^{|x|} x \otimes d_n(y).$$

This squares to zero as one checks directly (it is here that the introduction of some signs is necessary). Moreover, the symmetry isomorphism for the claimed symmetric monoidal structure also adds a sign: on elementary tensors, it is given by $\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x$. These sign conventions are often referred to as the *Koszul sign rule*. With these conventions, the tensor product of a chain complex of R-modules C with an R-module M is the same as the tensor product of C with the chain complex $M[0]$ consisting of M in degree 0 and 0's everywhere else.

4.51. **Definition** A short exact sequence of chain complexes of R-modules consists of chain maps

$$0 \to M'_\bullet \xrightarrow{i_\bullet} M_\bullet \xrightarrow{p_\bullet} M''_\bullet \to 0$$

such that the induced sequence of R-modules

$$0 \to M'_n \xrightarrow{i_n} M_n \xrightarrow{p_n} M''_n \to 0$$

is a short exact sequence in the sense of Remark 4.37 for every $n \in \mathbb{Z}$.

4.52. **Example** Let (X, A) be a pair of spaces. Then the sequence

$$0 \to C_\bullet^{\mathrm{sing}}(A) \to C_\bullet^{\mathrm{sing}}(X) \to C_\bullet^{\mathrm{sing}}(X, A) \to 0$$

is a short exact sequence of chain complexes. We have already see that the evident maps are maps of chain complexes, and by definition, for all $n \in \mathbb{Z}$, the sequence

$$0 \to C_n^{\mathrm{sing}}(A) \to C_n^{\mathrm{sing}}(X) \to C_n^{\mathrm{sing}}(X, A) \to 0$$

is short exact. Moreover, since all abelian groups appearing above are free for, we also obtain that for any abelian group M, the sequence

$$0 \to C_\bullet^{\mathrm{sing}}(A; M) \to C_\bullet^{\mathrm{sing}}(X; M) \to C_\bullet^{\mathrm{sing}}(X, A; M) \to 0$$

is a short exact sequence.

4.53. **Example** Let (X, A) be a pair of spaces and let

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of abelian groups. Then

$$0 \to C_\bullet^{\mathrm{sing}}(X, A; M') \to C_\bullet^{\mathrm{sing}}(X, A; M) \to C_\bullet^{\mathrm{sing}}(X, A; M'') \to 0$$

is a short exact sequence of chain complexes. This uses again that for all $n \in \mathbb{Z}$, the abelian groups $C_n^{\mathrm{sing}}(X, A)$ are free.

Short exact sequences of chain complexes play an important role.

4.54. **Lemma** *For a short exact sequence in* $\mathrm{Ch}(R)$

$$0 \to M'_\bullet \xrightarrow{i_\bullet} M_\bullet \xrightarrow{p_\bullet} M''_\bullet \to 0$$

and every $n \in \mathbb{Z}$ *there is defined a natural boundary map* $\partial_n \colon H_n(M''_\bullet) \to H_{n-1}(M_\bullet)$. *The associated sequence*

$$\dots H_n(M') \to H_n(M) \to H_n(M'') \xrightarrow{\partial_n} H_{n-1}(M') \to H_{n-1}(M) \to \dots$$

is long exact.

Proof. We apply the snake lemma Lemma B.32 three times (Exercise 1 Sheet 13). Once, to see that the upper and lower sequence in the following diagram are exact, and then a final time to the following diagram itself:

$$\begin{array}{ccccccc} & & M'_n/\mathrm{Im}(d'_{n+1}) & \longrightarrow & M_n/\mathrm{Im}(d_{n+1}) & \longrightarrow & M''_n/\mathrm{Im}(d''_{n+1}) \longrightarrow 0 \\ & & \downarrow{\scriptstyle d'_n} & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d''_n} \\ 0 & \longrightarrow & \ker(d'_{n-1}) & \longrightarrow & \ker(d_{n-1}) & \longrightarrow & \ker(d''_{n-1}) \end{array}$$

Then we use that the kernels of the vertical maps are isomorphic to $H_n(M')$, $H_n(M)$ and $H_n(M'')$, respectively and that the cokernels are isomorphic to $H_{n-1}(M')$, $H_{n-1}(M)$, and $H_{n-1}(M'')$. Naturality of the boundary operator follows from the naturality of the snake lemma. □

4.55. **Remark** Naturality of the bounday operator in Lemma 4.54 simply means that given a map f between two short exact sequences of chain complexes, i.e. a commutative diagram of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'_\bullet & \longrightarrow & M_\bullet & \longrightarrow & M''_\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'_\bullet} & & \downarrow{\scriptstyle f_\bullet} & & \downarrow{\scriptstyle f''_\bullet} & & \\ 0 & \longrightarrow & N'_\bullet & \longrightarrow & N_\bullet & \longrightarrow & N'_\bullet & \longrightarrow & 0 \end{array}$$

then for all $n \in \mathbb{Z}$, the square

$$\begin{array}{ccc} H_n(M''_\bullet) & \xrightarrow{\partial_n} & H_{n-1}(M'_\bullet) \\ {\scriptstyle H_n(f''_\bullet)}\downarrow & & \downarrow{\scriptstyle H_{n-1}(f'_\bullet)} \\ H_n(N''_\bullet) & \xrightarrow{\partial_n} & H_{n-1}(N'_\bullet) \end{array}$$

commutes.

4.56. **Corollary** *Let* (X, A) *be a pair of spaces and* $k \in \mathbb{Z}$ *an integer. Then there exists a natural map* $\beta_k \colon H_\bullet(X, A; \mathbb{Z}/k\mathbb{Z}) \to H_{\bullet-1}(X, A)$, *called the Bockstein operator. It sits inside a long exact sequence*

$$\dots \to H_n(X, A) \xrightarrow{\cdot k} H_n(X, A) \to H_n(X, A; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{\beta_n} H_{n-1}(X, A) \xrightarrow{\cdot k} H_{n-1}(X, A) \to \dots$$

4.57. **Remark** It follows that for each $n \in \mathbb{Z}$, we obtain a short exact sequence

$$0 \to H_n(X, A)/k \to H_n(X, A; \mathbb{Z}/k\mathbb{Z}) \to H_{n-1}(X, A)[k] \to 0$$

where the first and last term denote the cokernel and kernel of the multiplication by k map on the respective abelian group. Later, i.e. next term, we will show that

(1) this sequence splits, i.e. the middle term is the direct sum of the outer two terms, and
(2) there exists a variant for $\mathbb{Z}/k\mathbb{Z}$ replaced by an arbitrary abelian group M, in which the outer terms have to be replaced with terms involving the functors $-\otimes M$ and $\mathrm{Tor}(-,M)$.

Proof of parts of Theorem 4.34. (1) was already shown in Remark 4.24. For (3), we note that the evident map

$$\bigoplus_{i\in I} C_n^{\mathrm{sing}}(X_i) \to C_n^{\mathrm{sing}}(\coprod_{i\in I} X_i)$$

is a chain map on general categorical grounds. Furthermore, this map is in fact an isomorphism of chain complexes, and in particular induces an isomorphism on homology. (4) follows from Example 4.52 together with Lemma 4.54.

For (6), use the excision property (5) for the pair (X,B) with $U = X\setminus A$. Note that indeed $\overline{X\setminus A} = X\setminus \mathring{A} \subseteq \mathring{B}$ since $\mathring{A}\cup\mathring{B} = X$ and that we have $B\setminus(X\setminus A) = A\cap B$. We obtain a commutative diagram as follows.

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & H_{n+1}(A,A\cap B) & \longrightarrow & H_n(A\cap B) & \xrightarrow{i_*^A} & H_n(A) & \xrightarrow{p} & H_n(A,A\cap B) & \xrightarrow{\partial_n} & \dots \\
 & & \downarrow \cong & & \downarrow i_*^B & & \downarrow j_*^A & & \downarrow \cong & & \\
\dots & \longrightarrow & H_{n+1}(X,B) & \longrightarrow & H_n(B) & \xrightarrow{j_*^B} & H_n(X) & \xrightarrow{q} & H_n(X,B) & \longrightarrow & \dots
\end{array}$$

The map $\partial\colon H_n(X)\to H_{n-1}(A\cap B)$ is given by the composite

$$H_n(X)\to H_n(X,B) \xleftarrow{\cong} H_n(A,A\cap B) \xrightarrow{\partial_n} H_{n-1}(A\cap B).$$

It is then an exercise in diagram chasing that this map fits into a long exact sequence as claimed; let us show only some of the required things. For instance, assume $x\in\ker(\partial_n)$. Then its image in $H_n(A,A\cap B)$ lies in the kernel of $\partial_n\colon H_n(A,A\cap B)\to H_{n-1}(A\cap B)$ and hence is of the form $p(y)$ for some $y\in H_n(A)$ by exactness of the top row. Then $q(j_*^A(y)-) = 0$ and hence there exists $z\in H_n(B)$ such that $x = j_*^A(y) - j_*^B(z)$ as needed. Similarly, suppose that $x\in H_{n-1}(A\cap B)$ satisfies $i_*^A(x) = i_*^B(x) = 0$. Then there exists $z\in H_n(A,A\cap B)$ with $\partial_n(z) = x$ and its image in $H_n(X,B)$ lies in the kernel of the map to $H_{n-1}(B)$. Consequently, it lifts to $H_n(X)$, showing that x is in the image of the map $H_n(X)\to H_{n-1}(A\cap B)$. All other cases are similar or easier.

For (7), cover $\Sigma(X)$ by the contractible open subsets $\Sigma(X)\setminus\{N\}$ and $\Sigma(X)\setminus\{S\}$. The intersection is homotopy equivalent to X. The long exact Mayer-Vietoris sequence reads as

$$\dots\to H_n(\{S\})\oplus H_n(\{N\})\to H_n(\Sigma(X))\to H_{n-1}(X)\to H_{n-1}(\{S\})\oplus H_{n-1}(\{N\})\to\dots$$

from which the claim follows immediately. □

(7) above can also be proved using the following lemma (applied in the case $Y=*$) which we will need later.

4.58. **Lemma** *Let $f\colon X\to Y$ be a map. Then $C_\bullet^{\mathrm{sing}}(\mathrm{Cyl}(f),X)$ is canonically quasi-isomorphic to $C_\bullet^{\mathrm{sing}}(\mathrm{C}(f),*)$. In particular, there is a long exact sequence*

$$\dots\to H_n(X)\to H_n(Y)\to \widetilde{H}_n(\mathrm{C}(f))\to H_{n-1}(X)\to H_{n-1}(Y)\to\dots$$

Proof. Consider the pair $(\mathrm{C}(f),\mathrm{C}^+(X))$ where $\mathrm{C}^+(X)$ denotes the upper half part of the cone on X which is glued onto Y to form $\mathrm{C}(f)$. This pair is quasi-isomorphic to $(\mathrm{C}(f),*)$ since

$\mathrm{C}^+(X)$ is contractible. By excision, i.e. Theorem 4.34 (5), there is a further quasi-isomorphism $C_\bullet^{\mathrm{sing}}(\mathrm{C}(f), \mathrm{C}^+(X)) \cong C_\bullet^{\mathrm{sing}}(\mathrm{C}(f) \setminus \{\infty\}, \mathrm{C}^+(X) \setminus \{\infty\})$ where ∞ is the cone point of $\mathrm{C}^+(X)$. The latter is in turn quasi-isomorphic to the pair $(\mathrm{Cyl}(f), X)$ as needed. □

4.59. **Warning** We issue a warning about the naturality of the boundary operator in the Mayer-Viertoris long exact sequence. Note that we have in fact made a choice when defining it: Essentially, given A, B covering X, we consider the triple $(X, B, X \setminus A)$ and used excision for $X \setminus A \subseteq B$ to define the boundary map $\partial_n \colon H_n(X) \to H_{n-1}(A \cap B)$. To emphasize that we have chosen this triple (as opposed to the triple $(X, A, X \setminus B)$), let. us write more carefully $\partial_n^{A,B}$ for this map. Then, as just indicated, there is also a map $\partial_n^{B,A} \colon H_n(X) \to H_{n-1}(A \cap B)$, which uses excision for $X \setminus B \subseteq A$ instead. We will argue in Addendum 4.73 that we have $\partial_n^{B,A} = -\partial_n^{A,B}$. In particular, the boundary map in the Mayer-Vietoris long exact sequence is natural with respect to covers of X by *ordered* pairs (A, B). If one is uncareful about the order of (A, B) used to define the boudary map, then the diagrams involving ∂_n will only commute up to a sign. You will find such statements (diagrams of long exact Mayer-Viertoris sequences which commute possibly only up to a sign) in many places in the literature.

The properties we have left open are then homotopy invariance and excision. Both of these require a technical argument in chain complexes which we will have to provide. We will do so by proving a suitable abstract form of the "fundamental theorem of homological algebra" which in the literature is often referred to as (in similar form) the *theorem of acyclic models*. We first phrase the following general result and then specialize to the case we care about later. It will be convenient to introduce the following construction.

4.60. **Notation** Let $M \in \mathrm{Ch}(R)$ be a chain complex and let $n \in \mathbb{Z}$. Then we define a new chain complex $\sigma_{\leq n}(M)$ with $\sigma_{\leq n}(M)_k = M_k$ if $k \leq n$ and $\sigma_{\leq n}(M)_k = 0$ if $k > n$. There is an evident chain map $\sigma_{\leq n}(M) \to M$ which is the identity whenever possible.

4.61. **Lemma** *Let $\mathcal{A}$ be an abelian category, $P \in \mathrm{Ch}(\mathcal{A})$ a chain complex in $\mathcal{A}$ consisting of projective objects and let $M \in \mathrm{Ch}(\mathcal{A})$ be a further chain complex.*

(1) *Assume given $n \in \mathbb{Z}$ and a chain map $\sigma_{\leq n}(f) \colon \sigma_{\leq n}(P) \to \sigma_{\leq n}(M)$. Then $\sigma_{\leq n}(f)$ extends to a chain map $\sigma_{\leq n+1}(P) \to \sigma_{\leq n+1}(M)$ if and only if the canonical map $P_{n+1} \to H_n(M)$ induced by $f_n d_{n+1}^P$ is the zero map. Diagrammatically:*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_{n+1} & \xrightarrow{d^P_{n+1}} & P_n & \xrightarrow{d^P_n} & P_{n-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \cdots & \longrightarrow & M_{n+1} & \xrightarrow{d^M_{n+1}} & M_n & \xrightarrow{d^M_n} & M_{n-1} & \longrightarrow & \cdots \end{array}$$

(2) *Assume given $n \in \mathbb{Z}$, chain maps $f, g \colon P \to M$ and for $k \leq n-1$ maps $h_k \colon P_k \to M_{k+1}$ satisfying*

$$d_{k+1}^M h_k + h_{k-1} d_k^P = f_k - g_k.$$

Then there exists $h_n\colon P_n \to M_{n+1}$ satisfying the analogous relation if and only if the map $P_n \to H_n(M)$ induced by $f_n - g_n - h_{n-1}d_n^P$ is zero. Diagrammatically:

$$\begin{tikzcd} \dots \ar[r] & P_{n+1} \ar[r, "d^P_{n+1}"] \ar[d] & P_n \ar[r, "d^P_n"] \ar[d] \ar[dl, dashed, "h_n"'] & P_{n-1} \ar[r, "d^P_{n-1}"] \ar[d] \ar[dl, "h_{n-1}"'] & P_{n-2} \ar[r] \ar[d] \ar[dl, "h_{n-2}"'] & \dots \\ \dots \ar[r] & M_{n+1} \ar[r, "d^M_{n+1}"] & M_n \ar[r, "d^M_n"] & M_{n-1} \ar[r, "d^M_{n-1}"] & M_{n-2} \ar[r] & \dots \end{tikzcd}$$

In particular, given maps $f, g\colon P \to M$ such that $\sigma_{\leq n}(f) = \sigma_{\leq n}(g)$ and $\mathrm{Hom}_{\mathcal{A}}(P_k, H_k(M)) = 0$ for all $k \geq n+1$, then f and g are chain homotopic.

Proof. (1) Consider as written above the diagram

$$\begin{tikzcd} P_{n+1} \ar[r, "d^P_{n+1}"] \ar[d, dashed] & P_n \ar[r, "d^P_n"] \ar[d, "f_n"] & P_{n-1} \ar[d, "f_{n-1}"] \\ M_{n+1} \ar[r, "d^M_{n+1}"] & M_n \ar[r, "d^M_n"] & M_{n-1} \end{tikzcd}$$

Note that the composite $P_{n+1} \to P_n \to M_n \to M_{n-1}$ is zero by commutativity of the diagram and the fact that the top composite is zero. Therefore, we obtain a canonical map $P_{n+1} \to \ker(d_n^M)$ induced by f_n. The assumption that the map $P_{n+1} \to H_n(M)$ is zero implies that f_n in fact induces a map $P_{n+1} \to \mathrm{Im}(d^M_{n+1})$. Since $d_{n+1}\colon M_{n+1} \to \mathrm{Im}(d^M_{n+1})$ is surjective and P_{n+1} is projective, a dashed map f_{n+1} making the left square commute exists.

(2) Again, we note that

$$\begin{aligned} d_n^M(f_n - g_n - h_{n-1}d_n^P) &= d_n^M f_n - d_n^M g_n - (d_n^M h_{n-1})d_n^P \\ &= f_{n-1}d_n^P - g_{n-1}d_n^P - (f_{n-1} - g_{n-1} - h_{n-2}d^P_{n-1})d_n^P \\ &= 0 \end{aligned}$$

so that the assumptions imply that $f_n - g_n - h_{n-1}d_n^P$ induces a map $P_n \to \mathrm{Im}(d^M_{n+1})$. Hence, the projectivity of P_n again implies the existence of a map h_n with the required properties. The in particular is an immediate consequence of (2). □

Before coming to applications in topology of Lemma 4.61, we record the following important applications in homological algebra.

4.62. **Corollary** (The fundamental theorem of homological algebra) *Let $P \in \mathrm{Ch}(\mathcal{A})_{\geq 0}$ be a non-negatively graded chain complex consisting of projective objects in an abelian category $\mathcal{A}$ and let $M \in \mathrm{Ch}(\mathcal{A})$ be an exact chain complex. Suppose given $f_0\colon P_0 \to \ker(d_0) \subseteq M_0$. Then f_0 extends to a chain map $f\colon P \to M$ which is unique up to chain homotopy.*

Proof. By Lemma 4.61 (1), f_0 extends to a chain map $f\colon P \to M$ since $H_n(M) = 0$ for all $n \in \mathbb{Z}$ by assumption. Now suppose f and f' are two such extensions. For $k \leq 0$, define $0 = h_k\colon P_k \to M_{k+1}$. Then $d^M_{k+1}h_k + h_{k-1}d^P_k = f_k - f'_k$ holds true since $f_0 = f'_0$. Since M is exact, Lemma 4.61 (2) implies that there exists a chain homotopy between f and f'. □

4.63. **Corollary** *Any object in $\mathrm{Mod}(R)$ admits a projective resolution $P \in \mathrm{Ch}(R)_{\geq 0}$, unique up to chain homotopy equivalence and an injective resolution $I \in \mathrm{Ch}(R)_{\leq 0}$ again unique up to chain homotopy equivalence.*

Proof. Pick $M \in \mathrm{Mod}(R)$. Then there exists a surjection $P_0 \to M$ with P_0 surjective. Inductively, there exists a surjection $P_{n+1} \to \ker(P_n \to P_{n-1})$ with P_{n+1} projective. The resulting sequence

$$\dots P_{n+1} \to P_n \to P_{n-1} \to \dots \to P_0 \to 0$$

is a projective non-negatively graded chain complex P with $H_k(P) = 0$ for $k \neq 0$ and $H_0(P) \cong M$. Such a chain complex is called a projective resolution of M. It remains to show that such a projective resolution is unique up to chain homotopy equivalence. To see this, suppose P' is another such resolution. Then a dashed arrow in the diagram

$$\begin{array}{ccc} P_0 & \longrightarrow & H_0(P) \\ \downarrow & & \downarrow \cong \\ P'_0 & \longrightarrow & H_0(P') \end{array}$$

exists since P_0 is projective and the vertical lower map is surjective. By Lemma 4.61 (1), the obstructions to extending f_0 to a chain map $f\colon P \to P'$ vanish since $H_k(P') = 0$ for $k > 0$. Similarly, we can construct a chain map $g\colon P' \to P$ such that the composites fg and gf induce the identity on H_0. We will now show that given $\varphi\colon P \to P$ a chain self-map of a projective resolution of $H_0(P)$ which induces the identity on $H_0(P)$ is chain homotopic to the identity. To do so, we apply Lemma 4.61 (2): For $k < 0$ we have maps $h_k = 0$ satisfying all we want. Then we can construct h_0 since the composite $P_0 \to H_0(P)$ induced by $\mathrm{id} - \varphi_0$ is trivial. Inductively, we can then construct h_n for $n > 0$ simply because $H_n(P) = 0$ for $n > 0$.

The argument with injective resolutions is similar, or can formally be deduced from the above fact by arguing in $\mathrm{Mod}(R)^{\mathrm{op}}$. In this abelian category, projective objects are the injective objects of $\mathrm{Mod}(R)$. Therefore, in order to run the above argument, we only need to know that for every $M \in \mathrm{Mod}(R)$, there exists an injection $M \to I$ with I injective. This is a classical result which we will not argue here (perhaps we'll have it in the appendix at some point). □

4.64. **Remark** The argument above is an easy case of the following more general result which we will treat in Exercise 1 Sheet 14: Suppose P and Q are non-negatively graded chain complexes of projective modules and suppose given a quasi-isomorphism $f\colon P \to Q$. Then f is a chain homotopy equivalence. Note that for projective resolutions, the first step above produces a quasi-isomorphism between projective resolutions, and the second one then shows that this quasi-isomorphism is in fact a chain homotopy equivalence.

We move towards the topological applications of Lemma 4.61. First, we prove homotopy invariance of singular homology.

4.65. **Lemma** *Let X be a topological space and denote by i_0^X and i_1^X the evident inclusions $X \to X \times [0,1]$. Then $C_\bullet^{\mathrm{sing}}(i_0^X)$ and $C_\bullet^{\mathrm{sing}}(i_1^X)$ are chain homotopic, naturally in X. That is, there exists a chain homotopy h^X from $C_\bullet^{\mathrm{sing}}(i_0^X)$ to $C_\bullet^{\mathrm{sing}}(i_1^X)$ such that for any map $f\colon X \to Y$ and all $n \in \mathbb{Z}$, the diagram*

$$\begin{array}{ccc} C_n^{\mathrm{sing}}(X) & \xrightarrow{h_n^X} & C_{n+1}^{\mathrm{sing}}(X \times [0,1]) \\ {\scriptstyle C_n^{\mathrm{sing}}(f)}\downarrow & & \downarrow{\scriptstyle C_{n+1}^{\mathrm{sing}}(f \times \mathrm{id})} \\ C_n^{\mathrm{sing}}(Y) & \xrightarrow{h_n^Y} & C_{n+1}^{\mathrm{sing}}(Y \times [0,1]) \end{array}$$

commutes.

Proof. We will pretend that $\mathrm{Fun}(\mathrm{Top}, \mathrm{Mod}(\mathbb{Z}))$ is an abelian category (it only not so up to the set theoretic problem that it is not a category - for our applications, however, this does not play a role). Then $X \mapsto C^{\mathrm{sing}}_\bullet(X)$ is an object of $\mathrm{Ch}(\mathrm{Fun}(\mathrm{Top}, \mathrm{Mod}(\mathbb{Z})))$, which we claim to consist of levelwise projective objects. Indeed, $C^{\mathrm{sing}}_n(-) = \mathbb{Z}[\mathrm{Hom}_{\mathrm{Top}}(\Delta^n_{\mathrm{Top}}, -)]$. To see that this object is projective, consider a surjection $\alpha\colon F \to G$ in $\mathrm{Fun}(\mathrm{Top}, \mathrm{Mod}(\mathbb{Z}))$, i.e. a natural transformation of functors which is objectwise surjective, and a morphism $g\colon C^{\mathrm{sing}}_n(-) \to G$. To show projectivity, we need to show that there exists a dashed arrow in the diagram

$$\begin{array}{ccc} & & F \\ & \overset{f}{\nearrow} & \downarrow \\ C^{\mathrm{sing}}_n(-) & \longrightarrow & G \end{array}$$

making it commute. By definition of free abelian groups and the Yoneda lemma, the morphism g corresponds to an element $\hat{g} \in G(\Delta^n_{\mathrm{Top}})$ and the putative morphism f to an element $\hat{f} \in F(\Delta^n_{\mathrm{Top}})$. That the diagram commutes translates to the condition that $\alpha(\hat{f}) = \hat{g}$. Hence, the required f exists since α is objectwise surjective.

We will apply Lemma 4.61 (2) in the following way: We let $P = C^{\mathrm{sing}}_\bullet(-)$ and $M = C^{\mathrm{sing}}_\bullet(- \times [0,1])$ viewed as objects of $\mathrm{Ch}(\mathrm{Fun}(\mathrm{Top}, \mathrm{Ab}))$. For $n = 0$, we consider the map $h_0\colon C^{\mathrm{sing}}_0(-) \to C^{\mathrm{sing}}_1(- \times [0,1])$ which corresponds under the Yoneda lemma to canonical homeomorphism $\Delta^1 \to [0,1]$[15]. Then the composite

$$C^{\mathrm{sing}}_0(-) \to C^{\mathrm{sing}}_1(- \times [0,1]) \xrightarrow{d_1} C^{\mathrm{sing}}_0(- \times [0,1])$$

is given by $C^{\mathrm{sing}}_0(i_0) - C^{\mathrm{sing}}_0(i_1)$. We now want to argue inductively that h_0 can be extended to a natural chain homotopy between $C^{\mathrm{sing}}_\bullet(i_0)$ and $C^{\mathrm{sing}}_\bullet(i_1)$. This amounts to showing that for $n > 0$ certain maps $C^{\mathrm{sing}}_n(-) \to H_n(- \times [0,1])$ vanish. By Yoneda, such maps are equivalently described by elements in $H_n(\Delta^n_{\mathrm{Top}} \times [0,1])$ which is the trivial group by Proposition 4.29. □

4.66. **Remark** The naturality of the above chain homotopy implies (by the Yoneda lemma yet again) that the homotopies h^X_n are induced by elements $C^{\mathrm{sing}}_{n+1}(\Delta^n_{\mathrm{Top}} \times [0,1])$ satisfying certain relations. We have argued the existence of such elements by envoking the fact that $H_{n+1}(\Delta^n_{\mathrm{Top}} \times [0,1]) = 0$ for all $n \geq 0$ since $\Delta^n_{\mathrm{Top}} \times [0,1]$ is star-shaped, but we have not made explicit choices for such elements. This can, however, be done: Indeed, such elements are given by subdividing a prism $\Delta^n_{\mathrm{Top}} \times [0,1]$ into $(n+1)$-simplices, and then adding up (with appropriate signs) these $(n+1)$-simplices of $\Delta^n_{\mathrm{Top}} \times [0,1]$. To obtain the relevant $(n+1)$-simplices of $\Delta^n_{\mathrm{Top}} \times [0,1]$ denote for $k \in \{0, \dots, n\}$ and $\epsilon \in \{0,1\}$ by v^ϵ_k the point in $\Delta^n_{\mathrm{Top}} \times [0,1] \subseteq \mathbb{R}^{n+2}$ which has for $1 \leq i \leq n+1$ its $(k-1)$st coordinate equal to 1, its $n+2$nd coordinate equal to ϵ and all other components equal to 0. Then the convex hull of $\{v^0_0, \dots, v^0_i, v^1_i, \dots, v^1_n\}$ is an $(n+1)$-simplex inside $\Delta^n_{\mathrm{Top}} \times [0,1]$ which, added up (with signs determined by i) gives rise to the *Prism operator* $P_n\colon C^{\mathrm{sing}}_n(-) \to C^{\mathrm{sing}}_{n+1}(- \times [0,1])$, see e.g. [**?**] for the details.

[15]Concretely, h_0 sends a 0-simplex $\{x\} \subseteq X$ of a space X to the 1-simplex $\{x\} \times [0,1] \to X \times [0,1]$.

4.67. **Corollary** *Let $f, g\colon (X,A) \to (Y,B)$ be homotopic maps of pairs. Then $C^{\mathrm{sing}}_\bullet(f)$ and $C^{\mathrm{sing}}_\bullet(g)$ are chain homotopic maps $C^{\mathrm{sing}}_\bullet(X,A) \to C^{\mathrm{sing}}_\bullet(Y,B)$. In particular, for all $n \in \mathbb{Z}$, we have $H_n(f) = H_n(g)$.*

Proof. By Lemma 4.65 there exists for each $n \in \mathbb{Z}$ a commutative diagram

$$\begin{array}{ccccc} C^{\mathrm{sing}}_n(A) & \xrightarrow{h^A_n} & C^{\mathrm{sing}}_{n+1}(A \times [0,1]) & \longrightarrow & C^{\mathrm{sing}}_{n+1}(B) \\ \downarrow & & \downarrow & & \downarrow \\ C^{\mathrm{sing}}_n(X) & \xrightarrow{h^X_n} & C^{\mathrm{sing}}_{n+1}(X \times [0,1]) & \longrightarrow & C^{\mathrm{sing}}_{n+1}(Y) \end{array}$$

witnessing compatible chain homotopies between the maps induced on $C^{\mathrm{sing}}_\bullet(-)$ by the maps $f_{|A}, g_{|A}\colon A \to B$ and $f, g\colon X \to Y$. Passing to vertical cokernels, we obtain a map

$$C^{\mathrm{sing}}_n(X,A) \to C^{\mathrm{sing}}_{n+1}(Y,B)$$

which one checks to be a chain homotopy between the maps

$$C^{\mathrm{sing}}_\bullet(f), C^{\mathrm{sing}}_\bullet(g)\colon C^{\mathrm{sing}}_\bullet(X,A) \to C^{\mathrm{sing}}_\bullet(Y,B).$$

□

4.68. **Remark** Since for each abelian group M, the functor $-\otimes M\colon \mathrm{Ch}(\mathbb{Z}) \to \mathrm{Ch}(\mathbb{Z})$ preserves chain homotopies, one formally deduces that if $f, g\colon (X,A) \to (Y,B)$ are homotopic maps of pairs, then $H_n(f;M) = H_n(g;M)$ as maps $H_n(X,A;M) \to H_n(Y,B;M)$ for all $n \in \mathbb{Z}$.

To finish the verification of all properties of singular homology, it remains to prove excision. To do so, we need a procedure to make represent a given singular homology class by (sums of) simplices which are so small that they are contained in suitable subspaces of the given space. This motivates the following construction.

4.69. **Construction** Let $n \geq 0$ and let $\mathrm{bsd}_n \in C^{\mathrm{sing}}_n(\Delta^n_{\mathrm{Top}})$ be given inductively as follows. First, we define the barycenter $b_n = \frac{1}{n+1}(1,\dots,1) \in \Delta^n_{\mathrm{Top}}$ of Δ^n_{Top}. We note that it is a star for the starshaped (in fact convex) set Δ^n_{Top}. Now to the construction: We define $\mathrm{bsd}_0 = \mathrm{id}_{\Delta^0_{\mathrm{Top}}} \in C^{\mathrm{sing}}_0(\Delta^0_{\mathrm{Top}})$. Inductively, for $n \geq 1$ we then define

$$\mathrm{bsd}_n = \sum_{i=0}^{n} (-1)^i h_{n-1}[(\delta_i)_*(\mathrm{bsd}_{n-1})]$$

where we use $h_{n-1}\colon C^{\mathrm{sing}}_{n-1}(\Delta^n_{\mathrm{Top}}) \to C^{\mathrm{sing}}_n(\Delta^n_{\mathrm{Top}})$ as in Proposition 4.29 with star the barycenter b_n and $(\delta_i)_*\colon C^{\mathrm{sing}}_{n-1}(\Delta^{n-1}_{\mathrm{Top}}) \to C^{\mathrm{sing}}_{n-1}(\Delta^n_{\mathrm{Top}})$ is the map on singular $(n-1)$-simplices induced by $\delta_i\colon \Delta^{n-1}_{\mathrm{Top}} \to \Delta^n$.

4.70. **Lemma** (Barycentric subdivision) *The elements $\{\mathrm{bsd}_n\}_{n\geq 0}$ give rise to a chain map $\mathrm{bsd}\colon C^{\mathrm{sing}}_\bullet(-) \to C^{\mathrm{sing}}_\bullet(-)$ called the barycentric subdivision map. This map is naturally chain homotopic to the identity.*

Proof. By Yoneda, bsd_n gives rise to a map $C_n^{\mathrm{sing}}(-) \to C_n^{\mathrm{sing}}(-)$[16]. That this map is a chain map is equivalent means that for all $n \geq 0$, the diagram

$$\begin{array}{ccc} C_n^{\mathrm{sing}}(-) & \xrightarrow{\mathrm{bsd}_n} & C_n^{\mathrm{sing}}(-) \\ \downarrow d_n & & \downarrow d_n \\ C_{n-1}^{\mathrm{sing}}(-) & \xrightarrow{\mathrm{bsd}_{n-1}} & C_{n-1}^{\mathrm{sing}}(-) \end{array}$$

commutes. For $n = 0$ this is true tautologically, and for $n \geq 1$, by Yoneda yet again, this is equivalent to the equality

$$d_n(\mathrm{bsd}_n) = \sum_{i=0}^{n} (-1)^i (\delta_i)_*(\mathrm{bsd}_{n-1}) \in C_{n-1}^{\mathrm{sing}}(\Delta^n_{\mathrm{Top}}).$$

We then simply calculate

$$\begin{aligned} d_n(\mathrm{bsd}_n) &= \sum_{i=0}^{n} (-1)^i d_n h_{n-1}[(\delta_i)_*(\mathrm{bsd}_{n-1})] \\ &= \sum_{i=0}^{n} (-1)^i \Big[(\delta_i)_*(\mathrm{bsd}_{n-1}) - h_{n-2}(\delta_i)_*(d_{n-1}(\mathrm{bsd}_{n-1})) \Big] \\ &= \sum_{i=0}^{n} (-1)^i (\delta_i)_*(\mathrm{bsd}_{n-1}) - h_{n-2}\Big[\sum_{i=0}^{n} (-1)^i (\delta_i)_*(d_{n-1}(\mathrm{bsd}_{n-1})) \Big] \end{aligned}$$

where we have used the relation $d_n h_{n-1} = \mathrm{id} - h_{n-2} d_{n-1}$ established in the proof of Proposition 4.29. It then suffices to prove that

$$\sum_{i=0}^{n} (-1)^i \delta_i(d_{n-1}(\mathrm{bsd}_{n-1})) = 0.$$

For $n = 0, 1$ this is tautologically true. By induction, we then have

$$d_{n-1}(\mathrm{bsd}_{n-1}) = \sum_{j=0}^{n-1} (-1)^j \delta_j(\mathrm{bsd}_{n-1}).$$

Inserting this we obtain

$$\sum_{i=0}^{n} (-1)^i \delta_i(d_{n-1}(\mathrm{bsd}_{n-1})) = \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^{i+j} \delta_i \delta_j(\mathrm{bsd}_{n-2}) = 0$$

Indeed, the same argument as in Lemma 4.22 (proving $d^2 = 0$) shows that the latter terms cancel out as needed. To see that the resulting chain map bsd is homotopic to the identity, we note that its degree 0 part is the identity. As a consequence of Lemma 4.61 (2), there is no obstruction for constructing a natural chain homotopy between bsd and id. □

4.71. **Theorem** *Let (X, A) be a pair of spaces and let $U \subseteq A$ such that $\overline{U} \subseteq \mathring{A}$. Then for all $n \in \mathbb{Z}$, the canonical map $H_n(X \setminus U, A \setminus U) \to H_n(X, A)$ is an isomorphism.*

[16]Concretely, it sends $\sigma\colon \Delta^n_{\mathrm{Top}} \to X$ viewed as element in $C_n^{\mathrm{sing}}(X)$ to the element $\sigma_*(\mathrm{bsd}_n) \in C_n^{\mathrm{sing}}(X)$, i.e. it restricts σ to the barycentrically defined subsimplices in Δ^n and adds those up with appropriate signs.

Proof. Let us denote by $C_\bullet^{\mathrm{sing}}(X \mid X \setminus U, A)$ the image of the map of chain complexes

$$C_\bullet^{\mathrm{sing}}(X \setminus U) \oplus C_\bullet^{\mathrm{sing}}(A) \to C_\bullet^{\mathrm{sing}}(X)$$

and by $H_\bullet(X \mid X \setminus U, A)$ its homology. First, we note that the commutative diagram of chain complexes

$$\begin{array}{ccc} C_\bullet^{\mathrm{sing}}(A) \cap C_\bullet^{\mathrm{sing}}(X \setminus U) & \longrightarrow & C_\bullet^{\mathrm{sing}}(X \setminus U) \\ \downarrow & & \downarrow \\ C_\bullet^{\mathrm{sing}}(A) & \longrightarrow & C_\bullet^{\mathrm{sing}}(X \mid X \setminus U, A) \end{array}$$

induces an isomorphism on horizontal cokernels, as follows from the first isomorphism theorem in group theory[17]. Moreover, the top left term is simply given by $C_\bullet^{\mathrm{sing}}(A \setminus U)$. Next we will argue that the canonical inclusion $C_\bullet^{\mathrm{sing}}(X \mid X \setminus U, A) \to C_\bullet^{\mathrm{sing}}(X)$ is a quasi-isomorphism. Given this, for all $n \in \mathbb{Z}$, in the composite

$$H_n(X \setminus U, A \setminus U) \to H_n(C_\bullet^{\mathrm{sing}}(X \mid X \setminus U, A)/C_\bullet^{\mathrm{sing}}(A)) \to H_n(C_\bullet^{\mathrm{sing}}(X)/C_\bullet^{\mathrm{sing}}(A)) = H_n(X, A),$$

the first map is an isomorphism by the observation about horizontal cokernels in the above square of chain complexes, and the second map is an isomorphism by the 5-lemma.

Let us therefore show that the map $C_\bullet^{\mathrm{sing}}(X \mid X \setminus U, A) \to C_\bullet^{\mathrm{sing}}(X)$ is a quasi-isomorphism. To show surjectivty, let $\sigma \in C_n^{\mathrm{sing}}(X)$ represent an element of $H_n(X)$. By a Lebesgue lemma argument and the fact that the simplices in the barycentric subdivision of Δ^n_{Top} become successively smaller, there exists a $k \geq 1$ such that $\mathrm{bsd}^k(\sigma) \in C_n^{\mathrm{sing}}(X \mid X \setminus U, A)$. Here, we have used that $X \setminus \overline{U}, \mathring{A}$ is an open cover of X and applied the Lebesgue lemma to the open cover of Δ^n_{Top} given by σ^{-1} of this open cover. Since $C_\bullet^{\mathrm{sing}}(X \mid X \setminus U, A)$ is a subcomplex of $C_\bullet^{\mathrm{sing}}(X)$, we find that $\mathrm{bsd}^k(\sigma)$ represents an element of $H_n(X\, X \setminus U, A)$. Its image in $H_n(X)$ is represented by $\mathrm{bsd}^k(\sigma)$, but by Lemma 4.70, bsd induces the identity on homology, showing surjectivty of the map in question. Now assume that $\sigma \in C_n^{\mathrm{sing}}(X \mid X \setminus U, A)$ represents an element in homology whose image in $H_n(X)$ vanishes. Then there exists $\rho \in C_{n+1}^{\mathrm{sing}}(X)$ such that $d_{n+1}(\rho) = \sigma$. Similarly as above, there exists $k \geq 1$ such that $\mathrm{bsd}^k(\rho) \in C_{n+1}^{\mathrm{sing}}(X \mid X \setminus U, A)$. Moreover, $d_{n+1}(\mathrm{bsd}^k(\rho)) = \mathrm{bsd}^k(d_{n+1}(\rho)) = \mathrm{bsd}^k(\sigma)$. Now we note that $\mathrm{bsd}\colon C_\bullet^{\mathrm{sing}}(X) \to C_\bullet^{\mathrm{sing}}(X)$ restricts to a self map of $C_\bullet^{\mathrm{sing}}(X \mid X \setminus U, A)$ and that this map is again homotopic to the identity, which follows from the naturality of the map bsd and its chain homotopy to the identity. Consequently, we have $0 = [\mathrm{bsd}^k(\sigma)] = [\sigma] \in H_n(X \mid X \setminus U, A)$ and the map under investigation is injective. □

4.72. **Remark** You can go through the above argument and convince yourself that also for any abelian group M and all $n \in \mathbb{Z}$, the canonical map $H_n(X \setminus U, A \setminus U; M) \to H_n(X, A; M)$ is an isomorphism. Alternatively, one can use the following approach: Theorem 4.71 shows that $C_\bullet^{\mathrm{sing}}(X \setminus U, A \setminus U) \to C_\bullet^{\mathrm{sing}}(X, A)$ is a quasi-isomorphism. Since both chain complexes are levelwise projective and non-negatively graded, we can deduce from Remark 4.64 (see Exercise 1 Sheet 14) that it is in fact a chain homotopy equivalence. Hence it remains so after applying $- \otimes M$, and consequently, $H_n(X \setminus U, A \setminus U; M) \to H_n(X, A; M)$ is also an isomorphism.

[17]Recall that this says that for subgroups M, N of an abelian group A, there is a canonical isomorphism $(M + N)/N \cong M/(M \cap N)$.

4.73. **Addendum** We finish the argument in Warning 4.59 about the two boundary maps $H_n(X) \to H_{n-1}(A \cap B)$ for $A, B \subseteq X$ with $\mathring{A} \cup \mathring{B} = X$. To compare the two maps, we may represent an element of $H_n(X)$ by a cycle $x \in C_n^{\mathrm{sing}}(X)$. Using the barycentric subdivision as in the proof of Theorem 4.71, we may assume that $x = a + b$ where $a \in C_n^{\mathrm{sing}}(A)$ and $b \in C_n^{\mathrm{sing}}(B)$. Now let us recall the definition of $\partial_n^{A,B}$: It is given by the composite

$$H_n(X) \to H_n(X, B) \xleftarrow{\cong} H_n(A, A \cap B) \to H_{n-1}(A \cap B).$$

The first map sends $x = a+b$ simply to a which is already in the image of the map $C_n^{\mathrm{sing}}(A, A \cap B) \to C_n^{\mathrm{sing}}(X, B)$, and hence x is sent under the first two maps to the element represented by a. The final map is given by the boundary map in the long exact sequence of the pair $(A, A \cap B)$. Recall that this is concretely given choosing e.g. $a \in C_n^{\mathrm{sing}}(A)$ as a lift of its image in $C_n^{\mathrm{sing}}(A, A \cap B)$, applying d_n to it and observing that $d_n(a) \in C_n^{\mathrm{sing}}(A \cap B)$ is a cycle and hence represents an element in homology. Hence, we get

$$\partial_n^{A,B}(x) = [d_n(a)] \in H_{n-1}(A \cap B).$$

Analogously, we obtain

$$\partial_n^{B,A}(x) = [d_n(b)] \in H_{n-1}(A \cap B).$$

Therefore, we have

$$(\partial_n^{A,B} + \partial_n^{B,A})(x) = [d_n(a)] + [d_n(b)] = [d_n(a + b)] = [d_n(x)] = 0$$

since x is a cycle. As x was arbitrary, we deduce $\partial_n^{B,A} = -\partial_n^{A,B}$ as claimed.

4.74. **Proposition** *Theorem 4.34 is true for the functors $(X, A) \mapsto \{H_n(X, A; M)\}_{n\in\mathbb{Z}}$ for any abelian group M.*

Proof. (1) follows from the definitions: We have $C_n^{\mathrm{sing}}(*; M) = M$ for each $n \geq 0$ and the differentials are id and 0 in turn. (2) was argued in Remark 4.68 (3) follows formally from the fact that $- \otimes M$ is a left adjoint and hence commutes with coproducts. (4) follows from Example 4.52 and Lemma 4.54 just as before. (5) is the content of Remark 4.72, (6) formally follows from (5) and (4), just as we have proved (6) earlier. (7) formally follows from (6). □

4.75. **Lemma** *Let (X, A) be a pair of spaces. Then for all $n \in \mathbb{Z}$ there is a commutative triangle*

$$\begin{array}{ccc} H_n(X, A) & \longrightarrow & H_n(\Sigma(A), *) \\ & \overset{\partial}{\searrow} & \downarrow \sigma \\ & & H_{n-1}(A, *) \end{array}$$

whose diagonal arrow is the boundary map of the pair (X, A), whose vertical map is the suspension isomorphism and whose horizontal map is induced by the map $\mathrm{C}(i) \to \Sigma(A)$ under the isomorphism $H_n(X, A) \cong H_n(\mathrm{C}(i), *)$ from Lemma 4.58.

Proof. We consider the following commutative diagram of pairs of spaces:

$$\begin{array}{ccc} (\mathrm{C}(i), \mathrm{C}^+(A)) & \longrightarrow & (\Sigma(A), \mathrm{C}^+(A)) \\ \uparrow & & \uparrow \\ (\mathrm{Cyl}(i), A) & \longrightarrow & (\mathrm{C}^-(A), A) \end{array}$$

Here, we picture the vertical maps as inclusions, the left being where we attach $\mathrm{C}^+(A)$ on top of $\mathrm{Cyl}(i)$ to obtain $\mathrm{C}(i)$, and the right where $\mathrm{C}^-(A)$ is the southern hemisphere of the suspensions. Applying the functor H_n we obtain the top part of the following commutative square

$$\begin{array}{ccccc} & & H_n(\mathrm{C}(i), \mathrm{C}^+(A)) & \longrightarrow & H_n(\Sigma(A), \mathrm{C}^+(A)) \\ & & \cong\uparrow & & \uparrow\cong \\ H_n(X,A) & \xrightarrow{\cong} & H_n(\mathrm{Cyl}(i), A) & \longrightarrow & H_n(\mathrm{C}^-(A), A) \\ \downarrow\partial & & \partial\downarrow & & \downarrow\partial \\ H_{n-1}(A,*) & = & H_{n-1}(A,*) & = & H_{n-1}(A,*) \end{array}$$

of which the top to vertical maps are isomorphism by excision and the lower vertical maps are boundary maps for pairs. In particular, the lower square commutes by the naturality of the boundary map for pairs. Then we recall that the suspension isomorphism σ is by construction given by the right vertical composite. Finally we may use that $\mathrm{C}^+(A)$ is contractible so that the whole diagram gives a commutative square

$$\begin{array}{ccc} H_n(\mathrm{C}(i), *) & \longrightarrow & H_n(\Sigma(A), *) \\ \uparrow\cong & & \cong\downarrow\sigma \\ H_n(X,A) & \xrightarrow{\partial} & H_{n-1}(A,*) \end{array}$$

The claim then follows from the observation that the left vertical isomorphism in this square is indeed the one considered in Lemma 4.58. □

4.5. **Singular homology of CW complexes.** Having now verified all properties of singular homology, we can perform more calculations. First, we investigate the effect in homology of a cell attachement and general properties of homology of CW complexes. In the following, $H_n(-)$ refers to homology with arbitrary (but not in the notation fixed) coefficients.

4.76. **Corollary** *Consider a pushout diagram*

$$\begin{array}{ccc} \coprod_{i\in I} S^{n-1} & \longrightarrow & A \\ \downarrow & & \downarrow i \\ \coprod_{i\in I} D^n & \longrightarrow & X \end{array}$$

Then there exists a canonical exact sequence

$$0 \to H_n(A) \to H_n(X) \to \bigoplus_{i\in I} \widetilde{H}_{n-1}(S^{n-1}) \to H_{n-1}(A) \to H_{n-1}(X) \to 0$$

and $H_k(A) \to H_k(X)$ is an isomorphism for $k \neq n-1, n$. In particular, if Y is an n-dimensional CW complex, we have $H_k(Y) = 0$ for $k > n$ and $H_n(Y)$ is torsionfree. Moreover, for any CW complex Y, we have that the canonical map

$$\operatorname*{colim}_{n\geq 0} C_\bullet^{\mathrm{sing}}(\mathrm{sk}_n(Y)) \to C_\bullet^{\mathrm{sing}}(Y)$$

is an isomorphism. In particular, for each $k \geq 0$, the canonical map

$$\operatorname*{colim}_{n\geq 0} H_k(\mathrm{sk}_n(Y)) \to H_k(Y)$$

is an isomorphism.

Proof. Recall from Lemma 2.53 that $\mathrm{C}(i) \simeq X/A \cong \bigvee_{i\in I} S^n$. It then follows from Lemma 4.58 that $C^{\mathrm{sing}}_\bullet(X,A)$ is quasi-isomorphic to $\bigoplus_{i\in I} C^{\mathrm{sing}}_\bullet(S^n,*)$ since reduced homology commutes with (arbitrary) wedge sums of CW complexes (Exercise). Consequently, for Y an n-dimensional CW complex, we find inductively that $0 = H_k(\emptyset) \to H_k(Y)$ is an isomorphism for $k > n$. Moreover, considering $A = \mathrm{sk}_{n-1}(Y)$, the above exact sequence implies that $H_n(Y)$ is a subgroup of a free abelian group and hence itself free abelian. The first displayed isomorphism follows from the fact that Δ^n_{Top} is compact: By Proposition 2.35 (1) it follows that the image of any continuous map $\sigma\colon \Delta^n_{\mathrm{Top}} \to Y$ is contained in $\mathrm{sk}_n(Y)$ for some $n \geq 0$. The final part follows from the fact that a filtered colimit is an exact functor and hence commutes with taking homology (it preserves kernels and cokernels of maps). $\square$

4.77. **Example** In this example, we use the above to calculate $H_n(-)$ for all $n \geq 1$ for all the spaces we have previously calculated H_1, that is $\mathbb{RP}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$, T^2, K, Σ_g (notice that $T^2 = \Sigma_1$. We will revisit these calculations more systematically later.

We recall that there are pushouts

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & S^1 \vee S^1 \\ \downarrow & & \downarrow \\ D^2 & \longrightarrow & K \end{array} \qquad \begin{array}{ccc} S^1 & \xrightarrow{\alpha_g} & \bigvee_{2g} S^1 \\ \downarrow & & \downarrow \\ D^2 & \longrightarrow & \Sigma_g \end{array}$$

where $f = xyxy^{-1} \in \pi_1(S^1 \vee S^1)$ and $\alpha_g = [x_1,y_1] \cdot \dots \cdot [x_g,y_g]$. By Corollary 4.76 and the fact that $H_1(-)$ is abelian, we get an exact sequence

$$0 \to H_2(K) \to H_1(S^1) \xrightarrow{f_*} H_1(S^1 \vee S^1) \to H_1(K) \to 0$$

and the map f_* identifies with the map $(2,0)$ under the isomorphism $H_1(S^1 \vee S^1) \cong H_1(S^1) \oplus H_1(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$. This map is injective and has cokernel isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$. Hence, $H_2(K) \cong 0$ and $H_1(K) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$.

For Σ_g, we get an exact sequence

$$0 \to H_2(\Sigma_g) \to \mathbb{Z} \to \mathbb{Z}^{2g} \to H_1(\Sigma_g)$$

where now the middle map is induced by α_g and vanishes as a consequence of the Hurewicz theorem: the element $\alpha_g \in \pi_1(\bigvee S^1)$ lies in the commutator, i.e. the kernel of the map to the abelianization. Hence $H_2(\Sigma_g) \cong \mathbb{Z}$ and $H_1(\Sigma_g) \cong \mathbb{Z}^{2g}$.

We move on to the projective spaces and treat the cases $\mathbb{KP}^n$ for $\mathbb{K} = \mathbb{C}, \mathbb{H}$. Let k denote the real dimension of $\mathbb{K}$, i.e. 2 in case $\mathbb{K} = \mathbb{C}$ and 4 in case $\mathbb{K} = \mathbb{H}$. In Exercise 2 Sheet 6 we have seen that there are pushouts

$$\begin{array}{ccc} S^{nk-1} & \longrightarrow & \mathbb{KP}^{n-1} \\ \downarrow & & \downarrow \\ D^{nk} & \longrightarrow & \mathbb{KP}^n \end{array}$$

Consequently, by induction and the facts $\mathbb{CP}^1 \cong S^2$ and $\mathbb{KP}^1 \cong S^4$, we deduce that $H_k(\mathbb{CP}^n) = 0$ if k is odd, $H_{2k}(\mathbb{CP}^n) \cong \mathbb{Z}$ for $0 \leq k \leq n$, $H_k(\mathbb{CP}^n) = 0$ for $k > 2n$, and likewise $H_k(\mathbb{HP}^n) = 0$ unless $k = 4a$ with $0 \leq a \leq n$ in which case $H_{4a}(\mathbb{HP}^n) \cong \mathbb{Z}$.

Finally, we move to the more complicated case of $\mathbb{RP}^n$. Here, we have pushouts

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & \mathbb{RP}^{n-1} \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & \mathbb{RP}^n \end{array}$$

and $\mathbb{RP}^1 \cong S^1$. Since $H_k(\mathbb{RP}^n) = 0$ for $k > n$ we obtain an exact sequence

$$0 \to H_n(\mathbb{RP}^n) \to \mathbb{Z} \to H_{n-1}(\mathbb{RP}^{n-1}) \to H_{n-1}(\mathbb{RP}^n) \to 0$$

so we need to make the second map explicit. To do so, we can use Lemma 4.75 to see that this map is induced by the canonical map

$$S^n \cong \mathbb{RP}^n/\mathbb{RP}^{n-1} \to \Sigma\mathbb{RP}^{n-1}$$

under the suspension isomorphism. We may then consider the composite of this map with the projection $\Sigma\mathbb{RP}^{n-1} \to \Sigma(\mathbb{RP}^{n-1}/\mathbb{RP}^{n-2}) \cong S^n$. We will show later that the degree of the resulting self map of S^n is given by $1 + (-1)^n$, see Example 4.83.[18]

Inductively, we then deduce that

$$H_k(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } k = n \text{ and } n \text{ is odd} \\ \mathbb{Z}/2 & \text{if } 0 < k < n \text{ and } k \text{ is odd} \\ 0 & \text{else} \end{cases}$$

4.78. **Remark** Note that the same argument shows that $H_k(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ for all $0 \leq k \leq n$, compatible with the inductive proof of this result using Exercise 2 Sheet 14.

We have seen above that we can inductively calculate the homology of a cell attachment, provided we understand the effect of the attaching map on homology. This suggests that we should be able to calculate the homology of CW complexes in general. However, we have seen that concretely, we need to calculate degrees of self-maps of spheres in order to calculate homology. Often, the following local formula for the degree will allow us to do this.

4.79. **Definition** Let $f\colon S^n \to S^n$ be a map and $n \geq 1$. Let $x \in S^n$ and assume that f restricts to a homeomorphism $U \to V$ where U is an open neighborhood of x and V is an open neighborhood of $f(x)$. The composite

$$H_n(S^n) \to H_n(S^n, S^n\backslash\{x\}) \xleftarrow{\cong} H_n(U, U\backslash\{x\}) \xrightarrow{\cong} H_n(V, V\backslash\{f(x)\}) \to H_n(S^n, S^n\backslash\{f(x)\}) \xleftarrow{\cong} H_n(S^n)$$

is given by multiplication by a unique integer which we denote by $\deg_x(f)$, the local degree of f at x.

4.80. **Remark** All maps appearing in the definition of the local index are isomorphisms. Hence $\deg_x(f) \in \{\pm 1\}$.

4.81. **Proposition** *Let $f\colon S^n \to S^n$ be a map and $n \geq 1$. Assume that there exists $y \in S^n$ such that $f^{-1}(y) = \{x_1, \dots, x_k\}$ for some $k \geq 0$ and assume that for each $1 \leq i \leq k$, there*

[18] We will argue later that the map under investigation is the suspension of the map $S^{n-1} \to \mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2}$, where the first map is the canonical projection (which is also the attaching map of the n-cell of $\mathbb{RP}^n$). In Example 4.83 we calculate the degree of this unsuspended map, and hence also that of its suspension by Proposition 4.2.

exists an open neighborhood U_i of x and that f restricts to a homeomorphism $U_i \to V_i$ for some open neighborhood V_i of y. Then we have

$$\deg(f) = \sum_{i=1}^{k} \deg_{x_i}(f).$$

4.82. **Remark** Before giving the proof, we remark on the assumptions in Proposition 4.81. Without explaining in detail what this means, assume that f is a smooth map. Then a point $y \in S^n$ is called a regular value if for each $x \in f^{-1}(y)$, the differential at x is surjective (and hence an isomorphism for dimension reasons). In this case, the implicit function theorem implies that f restricts to a diffeomorphism in a neighborhood of x (onto a neighborhood of y). In particular, $f^{-1}(y)$ is discrete and compact (since it is a closed subset of the compact space S^n) and therefore finite. In other words, the assumptions in Proposition 4.81 are satisfied for all regular values $y \in S^n$. It is a theorem of Sard that regular values exist in abundance (more precisely, the set of regular values is dense in S^n). Moreover, any continuous map is homotopic to a smooth map, so the required data exist for in all situations we care about.

Proof. By passing to possibly smaller open sets, we may assume that all U_i are pairwise disjoint. Consider the triple $(S^n, S^n \setminus \{x_1, \dots, x_k\}, S^n \setminus \bigcup U_i)$. Then excision implies the isomorphism

$$H_n(S^n, S^n \setminus \{x_1, \dots, x_k\}) \cong H_n(\bigcup U_i, \bigcup (U_i \setminus \{x_i\})) \cong \bigoplus_{i=1}^{k} H_n(U_i, U_i \setminus \{x_i\})$$

where the last isomorphism follows from the fact that the U_i's are pairwise disjoint. Moreover, the map $H_n(S^n) \to H_n(S^n, S^n \setminus \{y\}) \cong H_n(V_i, V_i \setminus \{y\})$ is an isomorphism, and there is a commutative diagram

$$\begin{array}{ccc}
H_n(S^n) & \xrightarrow{\quad f_* \quad} & H_n(S^n) \\
\downarrow & & \downarrow \cong \\
H_n(S^n, S^n \setminus \{x_1, \dots, x_k\}) & \longrightarrow & H_n(S^n, S^n \setminus \{y\}) \\
\cong \uparrow & & \oplus \uparrow \\
\bigoplus_{i=1}^{k} H_n(U_i, U_i \setminus \{x_i\}) & \longrightarrow & \bigoplus_{i=1}^{k} H_n(V_i, V_i \setminus \{y\})
\end{array}$$

in which all so labelled morphisms are isomorphisms and the morphism labelled $\oplus$ is the one which is an isomorphism on each summand. Since the lower horizontal map is determined by the sum of the local degrees of f, the proposition follows. ☐

4.83. **Example** The degree of the map $f\colon S^n \to \mathbb{RP}^n \to \mathbb{RP}^n/\mathbb{RP}^{n-1} \cong S^n$ is $1 + (-1)^{n+1}$. Indeed, from its explicit description we see that N and S, the north and south pole of S^{n-1} are points as needed in order to apply Proposition 4.81. Hence we obtain

$$\deg(f) = \deg_N(f) + \deg_S(f) = \deg_N(f) + \deg_S(f \circ (-\mathrm{id}_{S^n}))$$

where the latter equality stems from the fact that $f = f \circ (-\mathrm{id}_{S^n})$. Now just as the degree, the local degree satisfies a compatibility formula in composites (when it makes sense, but in particular for compositions with homeomorphisms). In our case this reads as follows:

$$\deg_S(f \circ (-\mathrm{id}_{S^n})) = \deg_N(f) \cdot \deg_S(-\mathrm{id}_{S^n}).$$

Here, we have used $N = -S = -\mathrm{id}_{S^n}(S)$, of course. Proposition 4.81 and Proposition 4.2 give $\deg_S(-\mathrm{id}_{S^n}) = \deg(-\mathrm{id}_{S^n}) = (-1)^{n+1}$. Finally, we note that f restricted to the upper hemisphere $D^n_+ \subseteq S^n$ is concretely given by the canonical projection $D^n_+ \to D^n_+/\partial D^n_+ \cong S^n$. In particular, $\deg_N(f) = 1$, finishing the calculation.

Continuing towards calculating homology for general CW complexes, we make the following definition. In what follows, for a CW complex X, let us write $X_n := \mathrm{sk}_n(X)$ for its n-skeleton to shorten notation.

4.84. **Definition** For a CW complex X, an abelian group M, and $n \geq 0$, let $C^{\mathrm{cell}}_n(X; M) = H_n(X_n, X_{n-1}; M)$ and let $\partial^M_n \colon C^{\mathrm{cell}}_n(X; M) \to C^{\mathrm{cell}}_{n-1}(X; M)$ be the boundary map associated to the short exact sequence of chain complexes

$$0 \to C^{\mathrm{sing}}_\bullet(X_{n-1}, X_{n-2}; M) \to C^{\mathrm{sing}}_\bullet(X_n, X_{n-2}; M) \to C^{\mathrm{sing}}_\bullet(X_n, X_{n-1}; M) \to 0.$$

4.85. **Theorem** *Let X be a CW complex. Then $(C^{\mathrm{cell}}_\bullet(X; M), \partial^M_\bullet)$ is a chain complex, the* cellular chain complex *of X with coefficients in M. We have $C^{\mathrm{cell}}_\bullet(X; M) \cong C^{\mathrm{cell}}_\bullet(X; \mathbb{Z}) \otimes_{\mathbb{Z}} M$. Its homology $H^{\mathrm{cell}}_\bullet(X; M)$ is canonically*[19] *isomorphic to the singular homology $H_\bullet(X; M)$ of X. In particular, cellular homology does not depend on the choice of a CW structure.*

Proof. We claim that there is a commutative diagram

$$\begin{array}{ccc} H_n(X_n, X_{n-1}; \mathbb{Z}) \otimes_{\mathbb{Z}} M & \xrightarrow{\partial^{\mathbb{Z}}_n \otimes M} & H_{n-1}(X_{n-1}, X_{n-2}; \mathbb{Z}) \otimes_{\mathbb{Z}} M \\ \downarrow & & \downarrow \\ H_n(X_n, X_{n-1}; M) & \xrightarrow{\partial^M_n} & H_{n-1}(X_{n-1}, X_{n-2}; M) \end{array}$$

whose vertical maps are induced by sending $(\alpha \otimes m)$ to $m_*(\alpha)$ where we view $m \in M$ also as a map $m \colon \mathbb{Z} \to M$. As such, it induces a map $H_n(Y; \mathbb{Z}) \to H_n(Y; M)$ which we call m_*. One checks directly that this map is an isomorphism for $Y = S^n$, using that the suspension isomorphism is natural in coefficients. This reduces the claim to $n = 0$ where it is immediate. It follows that the above vertical maps induce a map $C^{\mathrm{cell}}_\bullet(X; \mathbb{Z}) \otimes_{\mathbb{Z}} M \to C^{\mathrm{cell}}_\bullet(X; M)$ which is levelwise an isomorphism.

Now we show that $\partial^{\mathbb{Z}}_n \partial^{\mathbb{Z}}_{n+1} = 0$, which by the above implies that $(C^{\mathrm{cell}}_\bullet(X; M), \partial^M_\bullet)$ is a chain complex for all M, and that this is isomorphic to $C^{\mathrm{cell}}_\bullet(X; \mathbb{Z}) \otimes_{\mathbb{Z}} M$. To do so, we observe that the above defined boundary maps ∂_n factor as the composite

$$H_n(X_n, X_{n-1}) \to H_{n-1}(X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2})$$

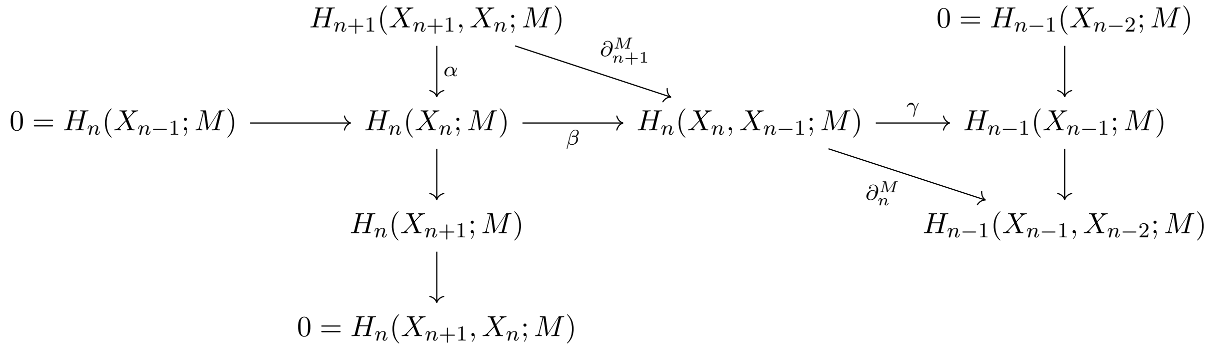
whose first map is the boundary map in the long exact sequence of the pair (X_n, X_{n-1}) and the second map is induced by the canonical projection. The composite $\partial^{\mathbb{Z}}_n \partial^{\mathbb{Z}}_{n+1}$ therefore factors through the composite

$$H_n(X_n) \to H_n(X_n, X_{n-1}) \to H_{n-1}(X_{n-1})$$

which is zero by exactness of the long exact sequence of pairs.

[19] More precisely, naturally in cellular maps.

Let us now consider the following commutative diagram consisting of exact rows and columns:



Then $\ker(\partial^M_n)/\mathrm{Im}(\partial^M_{n+1}) = H^{\mathrm{cell}}_n(X; M)$ by definition. All groups which are equal to 0 are so as a consequence of Corollary 4.76. Since the top most right term is zero, we have $\ker(\partial^M_n) = \ker(\gamma) = \mathrm{Im}(\beta) \cong H_n(X_n; M)$ since β is injective. Under this isomorphism, $\mathrm{Im}(\partial^M_{n+1})$ corresponds to $\mathrm{Im}(\alpha)$. Hence we deduce that $H^{\mathrm{cell}}_n(X; M) \cong H_n(X_n; M)/\mathrm{Im}(\alpha) \cong H_n(X_{n+1}; M)$ by exactness of the left vertical sequence. Moreover, the map $X_{n+1} \to X$ induces an isomorphism $H_{n+1}(X_n; M) \to H_{n+1}(X; M)$ by Corollary 4.76, finishing the proof of the theorem as all maps appearing in the above diagram are natural in cellular maps of CW complexes. □

4.86. **Remark** We recall that if X_n is obtained from X_{n-1} by attaching $|J_n|$ many cells, then $C^{\mathrm{cell}}_n(X; M) \cong \bigoplus_{J_n} M$ as we have argued in the proof of Corollary 4.76 for the case of $M = \mathbb{Z}$. The general case follows from the isomorphism $C^{\mathrm{cell}}_n(X; M) \cong C^{\mathrm{cell}}_n(X; \mathbb{Z}) \otimes_{\mathbb{Z}} M$. In particular, the cellular chain complex of a CW pair of finite type with coefficients in $\mathbb{Z}$ consists of finitely generated free modules in each degree, and is hence much smaller than the singular chain complex.

All of the following results can also be (in part, we have already done so) proven using Corollary 4.76.

4.87. **Corollary** *Let X be a CW complex with $X_n = X_{n-1}$, i.e. in which no n-cells are attached. Then $H_n(X; M) = 0$.*

Proof. We have $H_n(X; M) \cong H^{\mathrm{cell}}_n(X; M)$ which is a subquotient of $C^{\mathrm{cell}}_n(X; M)$ which vanishes. □

4.88. **Corollary** *Let X be an n-dimensional CW complex. Then $H_n(X; \mathbb{Z})$ is free abelian, in particular torsionfree.*

Proof. We have $H_n(X; \mathbb{Z}) \cong H^{\mathrm{cell}}_n(X; \mathbb{Z})$ which is a subgroup of the free abelian group $C^{\mathrm{cell}}_n(X; \mathbb{Z})$ since $C^{\mathrm{cell}}_{n+1}(X; \mathbb{Z})$ is trivial. The claim then follows from the fact that subgroups of free abelian groups are free abelian.[20] □

4.89. **Corollary** *Let X be a CW complex with the following property, for J_n the set of n-cells which are attached to X_{n-1}: If $J_n \neq \emptyset$ then $J_{n\pm 1} = \emptyset$. For instance, if X has only even dimensional or only odd dimensional cells (of positive dimension). Then for $n > 1$, we have $H_n(X; M) \cong \bigoplus_{J_n} M$.*

[20] For finitely generated groups this follows immediately from the classification, but the result is true in general, see e.g. [**?**].

Proof. The differentials in $C^{\mathrm{cell}}_\bullet(X;M)$ are trivial because their source or target is trivial. $\square$

4.90. **Example** Complex and Quaternionic projective spaces have the above property. Much more deep is the following wonderful result of Bott: Let G be a compact and simply connected Lie group. Then ΩG admits a CW structure with only even dimensional cells. In particular, $\pi_2(G) \cong \pi_1(\Omega G) \cong H_1(\Omega G) = 0$. Since any Lie group is homotopy equivalent to a compact Lie group (a maximal compact subgroup) and $\pi_2(G) \cong \pi_2(\widetilde{G})$ where $\widetilde{G}$ is a simply connected cover of G, in fact $\pi_2(G) = 0$ for all Lie groups.[21] There are various other proofs of this fact, and in general the topology of Lie groups is a wonderful topic.

We finish this part by describing the differentials in the cellular chain complex. To do so, let X be a CW complex and consider for $n \geq 2$ the composite

$$\varphi_n \colon \coprod_{J_n} S^{n-1} \to X_{n-1} \to X_{n-1}/X_{n-2} \simeq \bigvee_{J_{n-1}} S^{n-1}.$$

This map induces a map

$$(\varphi_n)_* \colon \bigoplus_{J_n} \mathbb{Z} \cong H_{n-1}(\coprod_{J_n} S^{n-1}) \to H_{n-1}(\bigvee_{J_{n-1}} S^{n-1}) \cong \bigoplus_{J_{n-1}} \mathbb{Z}.$$

with preferred isomorphism on source and target coming from the suspension isomorphism (and a once in life fixed choice of isomorphism $\widetilde{H}_0(S^0) \cong \mathbb{Z}$).

4.91. **Lemma** *For $n \geq 2$, the differential $\partial^{\mathbb{Z}}_n \colon C^{\mathrm{cell}}_n(X) \to C^{\mathrm{cell}}_{n-1}(X)$ is given by $(\varphi_n)_*$ under the isomorphisms $C^{\mathrm{cell}}_k(X) \cong \mathbb{Z}[J_k]$ discussed above. For $n = 1$, it is induced by sending a 1-cell σ to the difference of its boundaries.*

Proof. For $n = 1$, the differential is simply the boundary map $H_1(X_1, X_0) \to H_0(X_0) = \mathbb{Z}[X_0] = \mathbb{Z}[J_0]$ of the pair (X_1, X_0). The claim then follows from the fact that the 1-cells of X give rise to particular elements in $C^{\mathrm{sing}}_1(X_1)$ whose images in $C^{\mathrm{sing}}_1(X_1, X_0)$ define a basis for $H_1(X_1, X_0)$. On these particular elements, the boundary map indeed has the claimed form.

For $n \geq 2$, it follows from Lemma 4.75 that the differential is induced by the composite

$$\bigvee_{J_n} S^n \simeq X_n/X_{n-1} \to \Sigma(X_{n-1}) \to \Sigma(X_{n-1}/X_{n-2}) \simeq \bigvee_{J_{n-1}} S^n.$$

Since the degree is unchanged upon passing to suspended maps, it then suffices to argue that restricted to a single sphere, the first map is homotopic to the suspension of the attaching map $S^{n-1} \to X_{n-1}$. To show this, we may assume that X_n is obtained from X_{n-1} by attaching a single cell. In this case, the result follows from the naturality (up to homotopy) of the map $B/A \to \Sigma(A)$ for a cofibration $A \to B$: Indeed, we apply it to the square of vertical cofibrations

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\alpha} & X_{n-1} \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & X_n \end{array}$$

[21] A Lie group is a smooth manifold which is a topological group and all of whose structure maps (multiplication, inversion) are smooth maps. Examples are the usual matrix groups you know from linear algebra, like $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$, orthogonal and unitary groups as well as Spin and $\mathrm{Spin}^{\mathbb{C}}$ groups and various exceptional Lie groups. See e.g. [Kir08, Hal03] for introductions to Lie groups, their relation to Lie algebras and their classification and representation theory, and [DW98] for basic results on the topology of compact Lie groups.

which in turn gives a square (commutative up to homotopy)

$$\begin{array}{ccc} D^n/S^{n-1} & \xrightarrow{\cong} & \Sigma(S^{n-1}) \\ \downarrow & & \downarrow{\scriptstyle \Sigma(\alpha)} \\ X_n/X_{n-1} & \longrightarrow & \Sigma(X_{n-1}) \end{array}$$

where the vertical maps are as constructed in Corollary 2.54. □

4.92. **Example** For instance, we obtain that the cellular chain complex for $\mathbb{RP}^n$ is given as follows:

$$0 \to \mathbb{Z} \xrightarrow{1+(-1)^n} \mathbb{Z} \to \dots \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

which has homology as inductively calculated in Example 4.77 where the essential ingredient was also to identity a certain map via the local degree formula.

4.93. **Example** Similarly, there is a CW structure on $L(p; q_1, \dots, q_n)$, a $(2n-1)$-dimensional lens space with fundamental group C_p, having a single cell in each dimension $0 \le d \le 2n-1$ and where, as in the case of $\mathbb{RP}^n$, the projection maps $S^{2k-1} \to L(p; q_1, \dots, q_k)$ appear as attaching maps for the CW structure on $L(p; q_1, \dots, q_n)$ and the local degree formula allows to determine $C^{\text{cell}}_\bullet(L(p; q_1, \dots, q_n))$ to be given by:

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

In particular, $H_k(L(p; q_1, \dots, q_n))$ is independent of the choice of q_i's. This, however, is no coincidence: Any lens space $L(p; q_1, \dots, q_n)$ admits a map to a space called BC_p and this map induces an isomorphism on homology in degrees $0 \le d \le 2n-2$. Furthermore, the top dimensional homology is isomorphic to $\mathbb{Z}$ simply because $L(p; q_1, \dots, q_n)$ are orientable closed manifolds (we will see this homological behaviour of manifolds next term).

4.6. The Euler characteristic.

4.94. **Definition** Let X be a homologically finite topological space, i.e. such that $H_k(X)$ is finitely generated for all k and zero for all but finitely many k. We define its homological Euler characteristic $\chi(X)$ as follows:

$$\chi_{\text{hom}}(X) = \sum_{n \ge 0} (-1)^n \text{rk} H_n(X).$$

When X is a finite CW complex, we can also define its combinatorial Euler characteristic

$$\chi_{\text{CW}}(X) = \sum_{n \ge 0} (-1)^n |J_n|$$

where J_n denotes the set of n-cells of the CW structure.

4.95. **Theorem** *Let X be a finite CW complex and k any field. Then*

$$\chi_{\text{hom}}(X) = \chi_{\text{CW}}(X) = \sum_{n \ge 0} (-1)^n \dim_k(H_n(X; k))$$

i.e. the combinatorial and homological Euler characteristic agree and can be computed from homology with field coefficients for an arbitrary field.

Proof. Let $D = C^{\text{cell}}_\bullet(X;k) \cong C^{\text{cell}}_\bullet(X;\mathbb{Z}) \otimes_{\mathbb{Z}} k$. We will show

$$\sum(-1)^n \dim D_n = \sum(-1)^n \dim H_n(D).$$

Note that the LHS is equal to $\chi_{\text{CW}}(X)$ and that $H_n(D) \cong H_n(X;k)$ by Theorem 4.85. The above equality follows readily from the dimension formula for maps of k-vector spaces applied to all differentials in the chain complex:

$$\begin{aligned}\sum(-1)^n \dim D_n &= \sum(-1)^n[\dim\ker(d_n) + \dim\operatorname{Im}(d_n)]\\ &= \sum(-1)^n[\dim\ker(d_n) - \dim\operatorname{Im}(d_{n+1})]\\ &= \sum(-1)^n \dim H_n(D)\end{aligned}$$

as needed. This shows the second of the two equalities in the statement of the theorem. To see the first, we note that

$$\chi_{\text{hom}(X)} = \sum(-1)^n \dim H_n(X;\mathbb{Q})$$

and the latter term is by the first argument equal to $\chi_{\text{CW}}(X)$. □

4.96. **Remark** Note that Theorem 4.95 implies that the term $\sum(-1)^n \dim H_n(X;k)$ is independent of a chosen field k. The individual dimensions appearing, however, do very much depend on k: For instance, for all $n \geq 1$, we have $H_n(\mathbb{RP}^{2n};\mathbb{Q}) = 0$ whereas $H_n(\mathbb{RP}^{2n};\mathbb{F}_2) = \mathbb{F}_2$.

Consequently, we will from now on write $\chi(X)$ for the (unambigious) Euler characteristic of a finite CW complex. Note that the above says that the combinatorial Euler characteristic is an invariant under homotopy equivalences of CW complexes. An example of this kind is Euler's polyeder formula, which we phrase here as follows.

4.97. **Corollary** *Let Σ_g be an orientable surface of genus g which is triangulated by vertices, edges, and triangles. Then we find*

$$2 - 2g = |\textit{vertices}| - |\textit{edges}| + |\textit{triangles}|.$$

Proof. The LHS is $\chi_{\text{hom}}(\Sigma_g)$ and the RHS is $\chi_{\text{CW}}(\Sigma_g)$ for a CW structure associated to the triangulation. □

4.98. **Corollary** *Let X be a finite CW complex and $Y \to X$ an n-sheeted cover. Then $\chi(Y) = n \cdot \chi(X)$.*

Proof. In Exercise 1 Sheet 11 we have shown that Y admits a CW structure with $|J_n(Y)| = n \cdot |J_n(X)|$. □

4.99. **Corollary** *Let G be a finite group which acts freely (and hence covering-like) on S^{2n}. Then $|G| \leq 2$.*

Proof. We will not give a full proof. The main fact we will assume is that the quotient space S^{2n}/G admits a finite CW structure.[22] Then we find that $|G| \cdot \chi(S^{2n}/G) = \chi(S^{2n}) = 2$, proving the claim. □

Exercise. Give a proof of the above Corollary without making use of Euler characteristics.

[22] The quotient S^{2n}/G is a topological manifold. Unless the dimension of it is 4, it admits a CW structure (in dimension 4, the statement is not known) but in any case it is homotopy equivalent to a finite CW complex – this is in fact sufficient for our purposes and much easier than the existence of CW structures on the nose.

4.100. **Corollary** *Suppose given a covering map $\Sigma_h \to \Sigma_g$ between orientable surfaces. Then there is an $n \geq 1$ such that $h = ng - n + 1$.*

To finish, we apply Euler's polyeder formula and study platonic solids. Recall that a platonic solid is a convex polyhedron P embedded in $\mathbb{R}^3$ all of whose faces are congruent regular n-gons, all of whose vertices have the same number d of edges which touch the vertex, and where any two faces meet in precisely one edge (along which we think the surface of the polyhedron to be glued together). Note that with these assumptions and notations, we have $n, d \geq 3$.

4.101. **Corollary** *There are only 5 platonic solids: The cube, the tetrahedron, the octahedron, the icosahedron and the dodecahedron.*

Proof. A platonic solid P has a CW structure with faces given by regular polygons which are glued together at single edges. Let us denote by n the number of edges in such a face and d the number of edges that meet a fixed vertex. Let v be the number of vertices, e the number of edges and f the number of faces of the platonic solid. Then we find $nf = 2e = dv$ (each edge lies in exactly two faces and each edge meets exactly two vertices). Moreover, since the surface of the platonic solid is homotopy equivalent to a sphere, we obtain $v - e + f = 2$. Substituting $e = \frac{dv}{2}$ and $f = \frac{dv}{n}$, we obtain

$$\begin{aligned} 2 = v - e + f = v - \frac{dv}{2} + \frac{dv}{n} \\ \Leftrightarrow 4n = v(2n - dn + 2d) \\ \Rightarrow 0 < 2n - dn + 2d = -(n-2)(d-2) + 4 \\ \Leftrightarrow 4 > (n-2)(d-2) \end{aligned}$$

In total, we deduce that the only possible pairs for (n, d) are the pairs

- $(3, 3)$ which is the tetrahedron
- $(3, 4)$ which is the octahedron,
- $(3, 5)$ which is the icosahedron,
- $(4, 3)$ which is the cube, and
- $(5, 3)$ which is the dodecahedron

□

Appendix A. Category theory

A.1. **Definition** A category $\mathcal{C}$ consists of a class of objects $\mathrm{ob}(\mathcal{C})$, and for any two objects x and y a set $\mathrm{Hom}_{\mathcal{C}}(x, y)$ of morphisms between them, equipped with composition maps

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \times \mathrm{Hom}_{\mathcal{C}}(y, z) \to \mathrm{Hom}_{\mathcal{C}}(x, z)$$

and identities $* \to \mathrm{Hom}_{\mathcal{C}}(x, x)$ for all objects x, satisfying associativity and unitality. A morphism $f\colon c \to c'$ is called an isomorphism if there exists $g\colon c' \to c$ such that $gf = \mathrm{id}_c$ and $fd = \mathrm{id}_{c'}$. In this case we write f^{-1} for g (note that g is uniquely determined if its exists).

A category as defined above is sometimes also called a locally small category. A category is called small if the class of objects is a set. It is called essentially small if the class of isomorphism classes of objects is a set.

A.2. **Definition** Let $\mathcal{C}$ be a category. Then $\mathcal{C}^{\mathrm{op}}$ denotes the category with $\mathrm{ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{ob}(\mathcal{C})$ and

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(c, c') = \mathrm{Hom}_{\mathcal{C}}(c', c)$$

with same identities and obvious composition structure.

A.3. **Example** Sets, (abelian) groups, rings, vector spaces, modules, topological spaces all canonically form categories with morphisms set map, group homomorphisms, ring homomorphisms, linear maps, linear maps, and continuous maps.

A.4. **Example** Let $\mathcal{C}$ be a category with a single object $*$. Then $\mathcal{C}$ is equivalently described by the monoid (under composition) $\mathrm{End}_{\mathcal{C}}(*)$. Conversely, any monoid M gives rise to a category denoted BM which has one object $*$ and $M = \mathrm{End}_{BM}(*)$.

A.5. **Definition** A functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of a map $\mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$, $c \mapsto F(c)$, and for a pair of objects $c, c' \in \mathrm{ob}(\mathcal{C})$ a map $\mathrm{Hom}_{\mathcal{C}}(c, c') \to \mathrm{Hom}_{\mathcal{D}}(F(c), F(c'))$ which is

(1) compatible with identities: $F(\mathrm{id}_c) = \mathrm{id}_{F(c)} \in \mathrm{Hom}_{\mathcal{D}}(F(c), F(c))$, and
(2) compatible with composition: $F(g \circ f) = F(g) \circ F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(c), F(c''))$ for each pair of composable morphisms $c \xrightarrow{f} c' \xrightarrow{g} c''$.

A natural transformation between funtors $F, G\colon \mathcal{C} \to \mathcal{D}$ is a functor $\tau\colon \mathcal{C} \times [1] \to \mathcal{D}$ such that $\tau(-, 0) = F$ and $\tau(-, 1) = G$. Here, $[1]$ denotes the category with two objects 0 and 1 and exactly one non-identity morphism which goes from 0 to 1.

A.6. **Remark** Concretely, a natural transformation between F and G consists of a map $\tau_c\colon F(c) \to G(c)$ for all objects $c \in \mathrm{ob}(\mathcal{C})$ such that for each map $f\colon c \to c'$ the diagram

$$\begin{array}{ccc} F(c) & \xrightarrow{\tau_c} & G(c) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(c') & \xrightarrow{\tau_{c'}} & G(c') \end{array}$$

commutes. The maps τ_c are called the components of τ.

A.7. **Definition** A natural transformation $\tau\colon F \to G$ is called a natural isomorphism if there exists a natural transformation $\tau'\colon G \to F$ such that $\tau \circ \tau' = \mathrm{id}_G$ and $\tau' \circ \tau = \mathrm{id}_F$. Equivalently, if each map τ_c is an isomorphism (Exercise).

A.8. **Example** There is a category Cat whose objects are small categories and whose morphisms are functors. Note that the restriction to small categories is necessary, for if $\mathcal{C}$ is not small, the collection of functors from $\mathcal{C}$ to $\mathcal{D}$ form a proper class.

A category is called a groupoid if all its morphisms are isomorphisms. We denote by Gpd the full subcategory of Cat consisting of small groupoids.

A.9. **Example** Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

(1) There is a category $\mathcal{C} \times \mathcal{D}$ whose objects are pairs (c, d) with $c \in \mathcal{C}$ and $d \in \mathcal{D}$, and where $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((c,d),(c',d')) = \mathrm{Hom}_{\mathcal{C}}(c,c') \times \mathrm{Hom}_{\mathcal{D}}(d,d')$.
(2) If $\mathcal{C}$ is small, there is a category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ whose objects are functors and whose morphisms are natural transformations.

If $\mathcal{C}$ and $\mathcal{D}$ are groupoids, then so is $\mathcal{C} \times \mathcal{D}$. If $\mathcal{D}$ is a groupoid, then so is $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

A.10. **Example** (1) Let K be a field. Then association $V \mapsto V^\vee = \mathrm{Hom}_K(V, K)$ canonically is a functor $(\mathrm{Vect}_K)^{\mathrm{op}} \to \mathrm{Vect}_K$. The evaluation map $V \mapsto (V^\vee)^\vee$, $v \mapsto f \mapsto f(v)$ is a natural transformation $\mathrm{id} \to ((-)^\vee)^\vee$.
(2) Let G be a group, let $F, F' \colon BG \to Set$ be two functors. Then F and F' are equivalently described by the sets $M = F(*)$ and $M' = F'(*)$, which are acted upon by G. Indeed, part of the functor F is a map $G = \mathrm{End}_{BG}(*) \to \mathrm{End}_{\mathrm{Set}}(F(*))$ which is a monoid homomorphism (by the compatibility of functors with composition). A natural transformation $\tau \colon F \to F'$ is then the same thing as a G-equivariant map $f \colon M \to M'$, i.e. a map such that $f(gm) = gf(m)$ for all $g \in G$ and all $m \in M$ (Exercise).
(3) Consider the functor CRing $\to$ Groups from commutative rings to groups given by GL_n, $n \geq 1$, sending R to $\mathrm{GL}_n(R)$. Then the determinant is a natural transformation $\mathrm{GL}_n \to \mathrm{GL}_1$.
(4) The canonical maps $\pi_0(X) \to \pi(X)$ for topological spaces X are the components of a natural transformation $\pi_0 \to \pi$ as functors Top $\to$ Set.

A.11. **Definition** A functor $F \colon \mathcal{C} \to \mathcal{D}$ is called

(1) full, faithful, and fully faithful, respectively if for all pairs of objects $c, c' \in \mathrm{ob}(\mathcal{C})$ the induced map
$$\mathrm{Hom}_{\mathcal{C}}(c, c') \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(c), F(c'))$$
is surjective, injective, and bijective, respectively.
(2) essentially surjective, if for all $d \in \mathrm{ob}(\mathcal{D})$ there exists $c \in \mathrm{ob}(\mathcal{C})$ and an isomorphism $F(c) \cong d$.
(3) conservative, if F detects isomorphisms, that is, if f is an isomorphism if $F(f)$ is an isomorphism.

A.12. **Definition** A functor $F \colon \mathcal{C} \to \mathcal{D}$ is called an equivalence of categories if there exists $G \colon \mathcal{D} \to \mathcal{C}$ such that $FG \simeq \mathrm{id}_{\mathcal{D}}$ and $GF \simeq \mathrm{id}_{\mathcal{C}}$ (here, $\simeq$ refers to the existence of a natural isomorphism).

A.13. **Definition** A (co)limit of a functor $F \colon I \to \mathcal{C}$ consists of an object of $\mathcal{C}$, written $\mathrm{colim}_I F$ equipped with maps $F(i) \to \mathrm{colim}_I F$ for every i which are compatible in the sense that for every morphism $i \to j$ in I, the diagram

$$\begin{array}{ccc} F(i) & \longrightarrow & \operatorname{colim}_I F \\ \downarrow & \nearrow & \\ F(j) & & \end{array}$$

commutes (such a datum is also called a cone over F). This datum is required to satisfy the following universal property: Whenever given a further object $X \in \mathcal{C}$, also equipped with maps $F(i) \to X$ which are compatible in the above way, then there exists a unique morphism $\operatorname{colim}_I F \to X$ making the diagrams

$$\begin{array}{ccc} F(i) & \longrightarrow & X \\ \downarrow & \nearrow & \\ \operatorname{colim}_I F & & \end{array}$$

commute (we then say that this is a colimit cone).

Dually, a limit of F is an object $\lim_I F$, equipped with maps $\lim_I F \to F(i)$, which are again compatible, satisfying the dual universal property: Whenever we are given an object X equipped with compatible morphisms $X \to F(i)$ for all $i \in I$, there exists a unique morphism $X \to \lim_I F$ making the obvious diagram commute.

A.14. **Remark** Notice that such a universal property specifies an object up to unique isomorphism. Notice also that the universal property refers to more than just the object $\operatorname{colim}_I F$. The reference maps are part of the data, and this is what makes the object unique up to unique isomorphism.

A.15. **Definition** A category is called (co)complete, if it admits (co)limits indexed over arbitrary small categories. It is called bicomplete if it is both complete and cocomplete.

A.16. **Example**
(1) A colimit of the empty diagram $\emptyset \to \mathcal{C}$ is an initial object: It is an object which admits a unique morphism to any other object. Dually, A limit of the empty diagram $\emptyset \to \mathcal{C}$ is a terminal object: It is an object which admits a unique morphism from any other object.
(2) A colimit of the diagram $\bullet \leftarrow \bullet \rightarrow \bullet$ is called a pushout.
(3) A limit of the diagram $\bullet \rightarrow \bullet \leftarrow \bullet$ is called a pullback.
(4) A colimit over a discrete category (no non-identity morphisms) is called a coproduct.
(5) A limit over a discrete category is called a product.
(6) A colimit of the identity functor $\mathcal{C} \to \mathcal{C}$ is a terminal object (this is a great exercise!)

A.17. **Observation** One can phrase general (co)limits via inital and terminal objects. Given a functor $F\colon I \to \mathcal{C}$ we can consider the category of (co)cones of this functor. Given a category I we consider a new category $I^{\triangleleft}$ and $I^{\triangleright}$, which are constructed from I by adding an initial respectively a terminal object. There is an obvious functor $I \to I^{\triangleleft}$ and $I \to I^{\triangleright}$. We can thus consider the functor categories

$$\operatorname{Fun}_F(I^{\triangleleft}, \mathcal{C}) \text{ and } \operatorname{Fun}_F(I^{\triangleright}, \mathcal{C})$$

of functors which restrict to F along the above mentioned inclusion. These are called the categories of cones and cocones over F, respectively. A colimit is then an initial cone and a limit is a terminal cocone.

A.18. **Lemma** *Let $\mathcal{C}$ be a category which admits coproducts and coequalizers. Then $\mathcal{C}$ is cocomplete. Dually, when $\mathcal{C}$ admits products and equalisers, it is complete.*

Proof. The second part follows from the first by considering $\mathcal{C}^{\mathrm{op}}$. Let $F\colon I \to \mathcal{C}$ be a diagram. Then

$$\mathrm{Coeq}\Big[\coprod_{(f\colon i\to j)\in \mathrm{Arr}(\mathcal{C})} F(i) \rightrightarrows \coprod_{k\in \mathrm{ob}(\mathcal{C})} F(k)\Big]$$

is a colimit as one checks by the universal property. Here, the maps are as follows: Restricted to the component $F(i)$ indexed over $f\colon i \to j$, the one map is the canonical inclusion $F(i) \to \coprod_{k\in \mathrm{ob}(\mathcal{C})} F(k)$ and the other map is the map $F(f)\colon F(i) \to F(j)$ followed by the canonical inclusion to $\coprod_{i\in \mathrm{ob}(\mathcal{C})} F(i)$. □

A.19. **Lemma** *The category* Set *is bicomplete.*

The following lemma is immediate from the definition of (co)limits, and the just established fact established that the category Set is bicomplete.

A.20. **Lemma** *Let $\mathcal{C}$ be a category and let $F\colon I \to \mathcal{C}$ be an I-shaped diagram in $\mathcal{C}$. Then, for every object $x \in \mathcal{C}$, there are canonical bijections*

(1) $\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_I F, x) \cong \lim_I \mathrm{Hom}_{\mathcal{C}}(F(i), x)$, *and*
(2) $\mathrm{Hom}_{\mathcal{C}}(x, \lim_I F) \cong \lim_I \mathrm{Hom}_{\mathcal{C}}(x, F(i))$.

Moreover, this property characterizes (co)limits uniquely.

A.21. **Example** The category Cat has binary products given $(\mathcal{C}, \mathcal{D}) \mapsto \mathcal{C} \times \mathcal{D}$.

A.22. **Definition** An adjunction consists of a pair of functors $(F\colon \mathcal{C} \to \mathcal{D}, G\colon \mathcal{D} \to \mathcal{C})$ together with a natural isomorphism τ between the two functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Set}$ given by

$$\mathrm{Hom}_{\mathcal{D}}(F(-), -) \text{ and } \mathrm{Hom}_{\mathcal{C}}(-, G(-)).$$

If $f\colon F(X) \to Y$, then we refer to $\tau(f)\colon X \to G(Y)$ as the adjoint of f. We also refer to $\tau^{-1}(g)$ as the adjoint of $g\colon X \to G(Y)$.

A.23. **Remark** Equivalently, adjunctions can be described by unit and counit transformations satisfying the triangle equalities. In more detail, the maps $\eta_X\colon X \to GF(X)$ adjoint to the identity of $F(X)$ and the maps $\varepsilon_Y\colon FG(Y) \to Y$ adjoint to the identity of $G(Y)$ are the components of natural transformations $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ and $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$. η is called the unit of the adjunction and ε is called the counit of the adjunction. These maps satisfy the following relations, often called the triangle equalities: Namely, the composite

$$F(X) \xrightarrow{F(\eta_X)} F(GF(X)) \cong FG(F(X)) \xrightarrow{\eta_{F(X)}} F(X)$$

and the composite

$$G(Y) \xrightarrow{\eta_{G(Y)}} GF(G(Y)) \cong G(FG(Y)) \xrightarrow{G(\varepsilon_Y)} G(Y)$$

are the identities (this is somewhat of a fun exercise).

Given two functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ and natural transformations $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ and $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ satisfying the above triangle equalities, then there exists a unique natural isomorphism $\tau\colon \mathrm{Hom}_{\mathcal{D}}(F(-), -) \cong \mathrm{Hom}_{\mathcal{C}}(-, G(-))$ making F and G adjoint to each other such that the associated unit and counit are given by η and ε, respectively.

A.24. **Example** (1) Let $\mathcal{C}$ be a small category. Then the functors $\mathcal{C} \times -$ is left adjoint to the functor $\mathrm{Fun}(\mathcal{C}, -)$.
(2) Let K be a field. The forgetful functor $\mathrm{Vect}_K \to \mathrm{Set}$ has a left adjoint. It sends a set M to a vector space $\bigoplus_M K$ with basis given by M.
(3) The forgetful functor CRing $\to$ Set has a left adjoint. It sends a set M to a polynomial ring $\mathbb{Z}[X_m; m \in M]$ with one variable for each element in M.
(4) The inclusion functor Group $\to$ Monoid has both left and right adjoint (Exercise).
(5) The inclusion $\mathrm{td}-\mathrm{spaces} \subseteq \mathrm{Top}$ of totally disconnected spaces inside all topological spaces has a left adjoint (Exercise).

A.25. **Lemma** *Suppose given a functor $F\colon \mathcal{C} \to \mathcal{D}$. Specify for each object $d \in \mathrm{ob}(\mathcal{D})$ an object Gd together with a map $F(Gd) \to d$ such that the maps*

$$\mathrm{Hom}_{\mathcal{C}}(c, Gd) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(F(c), F(Gd)) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(c), d)$$

are isomorphisms. Then the association $d \mapsto Gd$ assembles into a functor $G\colon \mathcal{D} \to \mathcal{C}$ which is right adjoint to F. There is an obvious dual statement for the existence of left adjoints.

A.26. **Remark** The above condition is equivalently phrased as follows. Namely, that a functor $F\colon \mathcal{C} \to \mathcal{D}$ admits a right adjoint if and only if for all $d \in \mathcal{D}$, the functor $\mathrm{Hom}_{\mathcal{D}}(F(-), d)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is representable. I recommend trying to spell out why these two notions are equivalent.

A.27. **Lemma** *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor which admits right adjoints G and G'. Then there is a specified natural isomorphism between G and G'. (Adjoints, if they exist, are unique up to unique isomorphism).*

Proof. Consider the following two natural bijections

$$\mathrm{Hom}_{\mathcal{C}}(Gx, G'x) \cong \mathrm{Hom}_{\mathcal{D}}(FGx, x) \cong \mathrm{Hom}_{\mathcal{C}}(Gx, Gx).$$

Then the identity of Gx corresponds to a natural transformation $G \to G'$. Applying the same trick for $\mathrm{Hom}_{\mathcal{C}}(G'x, Gx)$ shows that this must be a natural isomorphism. □

A.28. **Lemma** *For an adjunction $(F, G, \eta, \varepsilon)$ we have*
(1) *F is fully faithful if and only if the unit η is an isomorphism,*
(2) *G is fully faithful if and only if the counit ε is an isomorphism,*
(3) *F is an equivalence of categories if and only if η and ε are isomorphisms.*

Moreover, F is an equivalence (with inverse G) if F is fully faithful and G is conservative (and vice versa).

A.29. **Lemma** *If $\mathcal{C}$ is bicomplete, then (co)lim is left/right adjoint to the constant diagram functor. In particular, forming (co)limits determines a functor*

$$\mathrm{Fun}(I, \mathcal{C}) \to \mathcal{C}.$$

Proof. Let's spell out the colimit case. Consider the constant functor $\mathrm{const}\colon \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$. Now we specify, for each functor $F\colon I \to \mathcal{C}$ an object, namely $\mathrm{colim}_I F$. Part of the datum of a colimit are compatible maps $\{F(i) \to \operatorname{colim}_I F\}_{\{i\in I\}}$ which are easily seen to assemble into a natural transformation

$$F \to \operatorname{const}(\operatorname*{colim}_I F).$$

Then we consider the composite

$$\operatorname{Hom}_{\mathcal{C}}(\operatorname*{colim}_I F, X) \to \operatorname{Hom}_{\operatorname{Fun}(I,\mathcal{C})}(\operatorname{const}(\operatorname*{colim}_I F), \operatorname{const}X) \to \operatorname{Hom}_{\operatorname{Fun}(I,\mathcal{C})}(F, \operatorname{const}X)$$

which is a bijection by the universal property of a colimit. The lemma thus follows from Lemma A.25. The case of limits is completely analogous. □

A.30. **Lemma** *Given an adjunction with $F\colon \mathcal{C} \to \mathcal{D}$ being left adjoint to $G\colon \mathcal{D} \to \mathcal{C}$, and given a further auxiliary small category I, then the functors*

$$F_*\colon \operatorname{Fun}(I,\mathcal{C}) \rightleftarrows \operatorname{Fun}(I,\mathcal{D})\colon G_*$$

again form an adjoint pair (with F_* left adjoint to G_*).

Proof. The adjunction is determined by a counit map $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ and a unit map $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ that satisfy the trinagle identities. We now use these to construct counit and unit maps for the pair of functors (F_*, G_*) as follows: Let $\varphi \in \operatorname{Fun}(I,\mathcal{D})$. We need to specify a natural map $\varepsilon_*\colon F_*(G_*(\varphi)) \to \varphi$ of functors $I \to \mathcal{D}$, so let $x \in \mathrm{E}$. We define the new counit ε_* to be the map

$$F(G(\varphi(x))) \xrightarrow{\varepsilon_{\varphi(x)}} \varphi(x).$$

It is easy to see that this is natural in φ, since ε itself is a natural transformation. Similarly we define a natural transformation $\eta_*\colon \psi \to G_*F_*(\psi)$ to be given by

$$\psi(y) \xrightarrow{\eta_{\psi(y)}} G(F(\psi(y))).$$

It is then easy to see that the snake identities are satisfied, because (ε, η) satisfy the snake identities. □

A.31. **Proposition** *Let $\mathcal{C}$ be a bicomplete category, then* $\operatorname{Fun}(I,\mathcal{C})$ *is bicomplete as well. A (co)limit of a diagram $X\colon J \to \operatorname{Fun}(I,\mathcal{C})$ is given by the functor sending $i \in I$ to* $\operatorname{colim}_J X(j)(i)$.

Proof. Let us argue that $\operatorname{Fun}(I,\mathcal{C})$ is cocomplete. The completeness argument is similar (or can be formally deduced from this case by applying op correctly). We claim that the composite

$$\operatorname{Fun}(J, \operatorname{Fun}(I,\mathcal{C})) \cong \operatorname{Fun}(I, \operatorname{Fun}(J,\mathcal{C})) \xrightarrow{\operatorname{colim}_J} \operatorname{Fun}(I,\mathcal{C})$$

is a colimit functor we wish to show exists. By Lemma A.30 this functor has a right given by

$$\operatorname{const}_*\colon \operatorname{Fun}(I,\mathcal{C}) \to \operatorname{Fun}(I, \operatorname{Fun}(J,\mathcal{C})) \cong \operatorname{Fun}(J, \operatorname{Fun}(I,\mathcal{C}))$$

it the proposition is shown once we convince ourselves that this is itself the constant functor (which is immediate from the definition), as then we allude to Lemma A.29. □

A.32. **Definition** Let $\mathcal{C}$ be a category. We denote the category of functors $\operatorname{Fun}(C^{\mathrm{op}}, \mathrm{Set})$ by $\mathcal{P}(\mathcal{C})$ and call it the category of presheaves on $\mathcal{C}$. An object $x \in \mathcal{C}$ determines a *representable* presheaf, namely the presheaf $\operatorname{Hom}_{\mathcal{C}}(-, x)$ which sends $y \in \mathcal{C}$ to the set of morphisms from y to x. This determines a functor $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ which is called the *Yoneda embedding.*

A.33. **Lemma** *The Yoneda lemma: Let* $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ *be a functor and* $x \in \mathcal{C}$ *an object. Then the map*

$$\mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(-,x), F) \to F(x)$$

given by sending η *to* $\eta(\mathrm{id}_x)$ *is a bijection. Moreover, the Yoneda embedding* $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ *is fully faithful.*

Proof. The inverse is given by sending an element $s \in F(x)$ to the function $\mathrm{Hom}_{\mathcal{C}}(y,x) \to F(y)$ sending f to $f^*(s)$. It is an explicit check to see that this is a natural transformation and an inverse the the above described map. The fully faithfulness follows immediately from the Yoneda Lemma: The effect of the Yoneda embedding on morphisms is the map

$$\mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(-,x), \mathrm{Hom}_{\mathcal{C}}(-,y))$$

given by sending f to

$$\mathrm{Hom}_{\mathcal{C}}(z,x) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(z,y).$$

We claim that this map is inverse to the map described in the Yoneda lemma, which is given by sending a map $f\colon \mathrm{Hom}_{\mathcal{C}}(x,y)$ to the function $\mathrm{Hom}_{\mathcal{C}}(z,x) \to \mathrm{Hom}_{\mathcal{C}}(z,y)$ given by sending φ to $\varphi^*(f) = f_*\varphi$. □

A.34. **Lemma** *Left adjoints preserve colimits, right adjoints preserve limits.*

Proof. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor which admits a right adjoint, say G. Let $X\colon I \to \mathcal{C}$ be a diagram which has a colimit $\mathrm{colim}_I X \in \mathcal{C}$. We claim that F sends that colimit to a colimit of the diagram $I \to \mathcal{C} \to \mathcal{D}$. In formulas, we claim that the canonical map $\mathrm{colim}_I F(X(i)) \to F(\mathrm{colim}_I X(i))$ induced from the compatible maps $F(X(i)) \to F(\mathrm{colim}_I X(i))$ that are part of the datum of the colimit (and then applying F) is an isomorphism. To see this, it suffices to show that it induces a bijection on hom sets for all other objects $y \in \mathcal{D}$:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}}(F(\operatorname*{colim}_I X(i)), y) &\cong \mathrm{Hom}_{\mathcal{C}}(\operatorname*{colim}_I X(i), Gy) \\
&\cong \lim_I \mathrm{Hom}_{\mathcal{C}}(X(i), Gy) \\
&\cong \lim_I \mathrm{Hom}_{\mathcal{D}}(F(X(i)), y) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(\operatorname*{colim}_I F(X(i)), y)
\end{aligned}$$

so we are done by the Yoneda lemma, see Lemma A.33. The argument for the claim that right adjoints preserve limits is similar. □

Appendix B. Some homological algebra

B.1. Modules.

B.1. **Definition** Let R be a ring. A (left) R-module consists of an abelian group M together with a ring map $R \to \mathrm{End}_{\mathbb{Z}}(M)$ written $r \mapsto (m \mapsto rm)$, and called the *scalar multiplication*. Equivalently, the scalar multiplication is determined by a map $R \times M \to M$, $(r, m) \mapsto rm$ satisfying the following axioms:

(1) $r(m + m') = rm + rm'$,
(2) $(r + r')m = rm + r'm$,
(3) $(rs)m = r(sm)$, and
(4) $1m = m$.

From now on, an R-module refers to a left R-module. A right R-module is then given by an R^{op}-module. If R is commutative, $R = R^{\mathrm{op}}$ and the notions agree. An R-submodule N of an R-module M is a subgroup $N \subseteq M$ closed under the scalar multiplication, that is: for $r \in R$ and $n \in N$, one has $rn \in N$. An R-module homomorphism $f \colon M \to M'$ between R-modules is a map of abelian groups, such that for all $r \in R$ and $m \in M$, one has $f(rm) = rf(m)$, i.e. that f is R-linear. We write $\mathrm{Hom}_R(M, N)$ for the set of R-linear maps from M to N and $\mathrm{Mod}(R)$ for the category of R-modules. Forgetting the scalar multiplication and the abelian group structure gives forgetful functors $\mathrm{Mod}(R) \to \mathrm{Ab} \to \mathrm{Set}$.

B.2. **Example** (1) Let K be a field. Then a K-module is precisely a K-vector space.

(2) Let R be a ring. Then R is an R-module via the multiplication map of R. An R-submodule of R is precisely a left ideal of R.
(3) Let $f \colon S \to R$ be a ring map. Then there is a canonical restriction of scalars functor $\mathrm{Mod}(R) \to \mathrm{Mod}(S)$, sending an R module $(M, R \to \mathrm{End}_{\mathbb{Z}}(M))$ to the S-module $(M, S \to R \to \mathrm{End}_{\mathbb{Z}}(M))$. In particular, R is canonically an S-module with module multiplication given by $s \cdot r = f(s)r$.
(4) The forgetful functor $\mathrm{Mod}(R) \to \mathrm{Ab}$ is conservative (that is, it detects isomorphisms). Indeed, if $f \colon M \to N$ is R-linear and bijective, then its inverse f^{-1} satisfies
$$f(f^{-1}(rn)) = rn = r(ff^{-1}(n)) = f(rf^{-1}(n))$$
so that the bijectivity of f implies that $f^{-1}(rn) = rf^{-1}(n)$.
(5) The forgetful functor $\mathrm{Mod}(\mathbb{Z}) \to \mathrm{Ab}$ is an isomorphism of categories (Exercise). Under this isomorphism, the forgetful functor $\mathrm{Mod}(R) \to \mathrm{Ab}$ corresponds to the restriction of scalars functor $\mathrm{Mod}(R) \to \mathrm{Mod}(\mathbb{Z})$ along the unique map of rings $\mathbb{Z} \to R$.
(6) Given an R-linear map $f \colon M \to N$, the kernel of f, $\ker(f) = \{m \in M \mid f(m) = 0\}$ is an R-submodule of M and the Image $\Im(f) = f(M) \subseteq N$ is canonically an R-submodule of N.
(7) Given an R-module M and an R-submodule $N \subseteq M$, the quotient of abelian groups M/N is canonically an R-module vie $r[m] = [rm]$. It satisfies the expected universal property: For any other R-module L, the quotient map induces an injection
$$\mathrm{Hom}_R(M/N, L) \longrightarrow \mathrm{Hom}_R(M, N)$$
whose image consists precisely of those R-linear maps $f \colon M \to N$ whose kernel is contained in N.
(8) Given an R-linear map $f \colon M \to N$, we define its cokernel $\mathrm{coker}(f)$ to be the quotient R-module $N/\mathrm{Im}(f)$. Kernel and cokernel then have the expected universal properties:

the maps

$$\mathrm{Hom}_R(L,\ker(f)) \to \mathrm{Hom}_R(L,M) \quad \text{and} \quad \mathrm{Hom}_R(\mathrm{coker}(f),L) \to \mathrm{Hom}_R(N,L)$$

are injective with image given by those maps $L \to M$ or $N \to L$ respectively, whose composite with f is the zero map.

(9) Given a commutative ring R, an R-module M and an ideal $\mathfrak{a} \subseteq R$, we let $\mathfrak{a}M = \{\sum_i a_i m_i \mid a_i \in \mathfrak{a} \text{ and } m_i \in M\}$. This is an R-submodule of M. Then quotient R-module $M/\mathfrak{a}M$ is in the image of the restriction of scalars functor $\mathrm{Mod}(R/\mathfrak{a}) \to \mathrm{Mod}(R)$, that is, the scalar multiplication map $R \to \mathrm{End}_{\mathbb{Z}}(M/\mathfrak{a}M)$ factors through the projection $R \to R/\mathfrak{a}$.

(10) Given a family of R-modules $\{M_i\}_{i\in I}$ indexed over a set I, then the abelian groups $\bigoplus_i M_i$ and $\prod_i M_i$ canonically admit the structure of R-modules via componentwise scalar multiplication. We have that $\bigoplus_i M_i \subseteq \prod_i M_i$ is an R-submodule. These constructions are coproducts and products in the categorical sense, that is, they satisfy the following universal properties: For each $j \in I$, the canonical maps $M_j \to \bigoplus_i M_i$ and $\prod_i M_i \to M_j$ are R-linear and or another R-module N, one has that the canonical maps

$$\mathrm{Hom}_R(\bigoplus_i M_i, N) \to \prod_i \mathrm{Hom}_R(M_i,N) \quad \text{and} \quad \mathrm{Hom}_R(N,\prod_i M_i) \to \prod_i \mathrm{Hom}_R(N,M_i)$$

are bijections. We write $R^{(I)}$ for $\bigoplus_I R$ and R^I for $\prod_I R$. Modules isomorphic to $R^{(I)}$ are called *free* (on the set I), and *finitely generated free* if I is finite (in which case $R^{(I)} \cong R^I$).

(11) Given a commutative ring R and R-modules M and N, the set of R-linear maps $\mathrm{Hom}_R(M,N)$ is naturally an R-module[23]. Indeed, first we note that it is an abelian group with monoid structure given as follows[24]. For $f,g \in \mathrm{Hom}_R(M,N)$, define the map $f+g$ via $(f+g)(m) = f(m)+g(m)$. Immediately from the definitions, we find that $f+g \in \mathrm{Hom}_R(M,N)$. The neutral element is the zero map 0 sending all elements of M to 0. The inverse of a map f is then given by $-f$, defined via $(-f)(m) = -f(m)$. This shows that $\mathrm{Hom}_R(M,N)$ is indeed an abelian group. We define a scalar multiplication as follows: For $r \in R$ and $f \in \mathrm{Hom}_R(M,N)$ we set $(rf)(m) = rf(m)$. Since R is commutative, one checks that rf is again R-linear and that this defines an R-module structure on $\mathrm{Hom}_R(M,N)$.

(12) Given a commutative ring R and an R-linear map $f\colon M \to M'$ and another R-module N, the canonical maps

$$\mathrm{Hom}_R(N,M) \xrightarrow{f_*} \mathrm{Hom}_R(N,M') \quad \text{and} \quad \mathrm{Hom}_R(M',N) \xrightarrow{f^*} \mathrm{Hom}_R(M,N)$$

given by postcomposition and precomposition with f respectively, are R-linear. In particular, the bijections appearing in the display in (10) are isomorphisms of R-modules. These maps induce functors

(a) $\mathrm{Hom}_R(M,-)\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R)$, and
(b) $\mathrm{Hom}_R(-,M)\colon \mathrm{Mod}(R)^{\mathrm{op}} \to \mathrm{Mod}(R)$,

which can in fact be combined to a single functor $\mathrm{Hom}_R(-,-)\colon \mathrm{Mod}(R)^{\mathrm{op}} \times \mathrm{Mod}(R) \to \mathrm{Mod}(R)$. Composing this functor with the forgetful functor $\mathrm{Mod}(R) \to \mathrm{Set}$ gives the usual Hom functor of the category $\mathrm{Mod}(R)$.

[23] The same is not generally true if R is not commutative.

[24] This is true also when R is not commutative.

Exercise. The category $\mathrm{Mod}(R)$ admits all small limits and colimits and the restriction of scalar functors $\mathrm{Mod}(R) \to \mathrm{Mod}(S)$, for ring maps $S \to R$, commute with all small limits and colimits. Hint: It suffices to consider equalizers and coequalizers.

Exercise. Let R be a commutative ring and M be an R-module. Then the scalar multiplication map $R \to \mathrm{End}_{\mathbb{Z}}(M)$ factors through the forgetful map $\mathrm{End}_R(M) \to \mathrm{End}_{\mathbb{Z}}(M)$, that is, scalar multiplication by a fixed element of R on M is an R-linear map.

B.3. **Definition** An R-module M is called

(1) *finitely generated*, if there exists a finite set I and a surjection $R^I \to M$. In other words, if M is a quotient of a finitely generated free R-module,
(2) *finitely presented*, if there exists a finite set I and a surjection $R^I \to M$ whose kernel is again a finitely generated R-module.

In what follows we will discuss tensor products of R-modules which we do for convenience in the case where R is commutative (so that we do not have to talk about left *and* right modules at the same time).

B.4. **Definition** Let R be a commutative ring and M, N and L be R-modules. A map $f\colon M \times N \to L$ is called *R-bilinear* if for all $m, n' \in M$, $n, n' \in N$ and $r \in R$, one has:

(1) $f(m + m', n) = f(m, n) + f(m', n)$,
(2) $f(m, n + n') = f(m, n) + f(m, n')$, and
(3) $f(rm, n) = rf(m, n) = f(m, rn)$.

We denote by $\mathrm{Hom}_{R,R}(M \times N, L)$ the set of R-bilinear maps. Note again, that it is canonically an R-module via $(rf)(m, n) = r \cdot f(m, n)$.

B.5. **Remark** For R-modules M, N and L, the map

$$\mathrm{Hom}_{R,R}(M \times N, L) \to \mathrm{Hom}_R(N, \mathrm{Hom}_R(M, L))$$

sending f to the map $n \mapsto f(-, n)$ is R-linear and a bijection, hence an isomorphism of R-modules.

B.6. **Definition** Let R be a commutative ring and let M and N be R-modules. A tensor product of M and N consists of an R-module $M \otimes_R N$ equipped with a R-bilinear map $M \times N \to M \otimes_R N$ satisfying the following universal property: For every R-bilinear map $\varphi\colon M \times N \to L$, there exists a unique R-linear map $\bar{\varphi}\colon M \otimes_R N \to L$ making the diagram

$$\begin{array}{ccc} M \times N & \xrightarrow{\varphi} & L \\ \downarrow & \nearrow_{\bar{\varphi}} & \\ M \otimes_R N & & \end{array}$$

commute. In other words, the universal property says that the canonical map

$$\mathrm{Hom}_R(M \otimes_R N, L) \longrightarrow \mathrm{Hom}_{R,R}(M \times N, L)$$

is a bijection.

B.7. **Remark** If a tensor product exists, it is specified up to unique isomorphism by its universal property. The question thus really is, do tensor products exist. The answer is yes:

B.8. **Lemma** *Let R be a commutative ring and M and N be R-modules. Then a tensor product $(M \otimes_R N, M \times N \to M \otimes_R N)$ exists.*

Proof. We define $M \otimes_R N$ by brut-force: First we consider $F(M,N) = R^{(M\times N)}$, the free R-module on the set $M \times N$. This comes with a map of sets $\iota\colon M \times N \to F(M,N)$. The universal property says that the map $\varphi\colon M \times N \to L$ extends uniquely to a map $\tilde{\varphi}\colon F(M,N) \to L$ of R-modules. The map ι is not R-bilinear: For instance, $\iota(rm,n) \neq r\iota(m,n)$, and likewise $\iota(m,n+n') \neq \iota(m,n) + \iota(m,n')$. So consider the sub R-module V of $F(M,N)$ generated by the set

$$\Big\{\iota(m+m',n)-\iota(m,n)-\iota(m',n), \iota(m,n+n')-\iota(m,n)-\iota(m,n'), \iota(rm,n)-r\iota(m,n), \iota(m,rn)-r\iota(m,n)\Big\}$$

where $m,m' \in M, n,n' \in N, r \in R$ are arbitrary elements. Then define $M \otimes_R N$ as the quotient R-module $F(M,N)/V$. By construction, the composite

$$M \times N \xrightarrow{\iota} F(M,N) \longrightarrow F(M,N)/V = M \otimes_R N$$

is R-bilinear. Moreover, since φ is R-bilinear, the map $\tilde{\varphi}$ extends uniquely to the quotient $M \otimes_R N$, showing that this object satisfies the required universal property. □

B.9. **Remark** The image under the map $M \times N \to M \otimes_R N$ of an element (m,n) is often written $m \otimes n$ and called an *elementary tensor*. It is important to keep in mind that not all elements of $M \otimes_R N$ are of this form (but they form a generating set, that is, every element is of a sum of elementary tensors). For instance, $m \otimes n + m' \otimes n'$ is in general not an elementary tensor. But the tensor product is bilinear, so that

$$m \otimes n + m \otimes n' = m \otimes (n + n') \quad \text{and} \quad m \otimes n + m' \otimes n = (m + m') \otimes n.$$

The R-module structure is then given by $r \cdot (m \otimes n) = rm \otimes n = m \otimes rn$, that is, we are allowed to move scalars from R through the tensor sign.

B.10. **Example** Let p and q be different prime numbers. Then $\mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Z}/q\mathbb{Z} = 0$. Indeed, it suffices to show that any biadditive map $f\colon \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \to M$, for an arbitrary abelian group M, is the zero map. Since $f(m,n) = mn \cdot f(1,1)$, this map is determined by $x = f(1,1)$. Moreover, this element satisfies $px = qx = 0$ since $pf(1,1) = f(p,1) = f(0,1) = 0$ and likewise $qf(1,1) = f(1,q) = f(1,0) = 0$. But since p and q are coprime, there exists n,m such that $np + mq = 1$, and consequently that

$$x = (np + mq)x = npx + mqx = 0.$$

B.11. **Example** Let p be a prime number. Then $\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. Indeed, by the same argument as above, any biadditive map $f\colon \mathbb{Z}/p\mathbb{Z} \times \mathbb{Q} \to M$ is determined by $m = f(1,1)$. Then we have

$$m = pf(1,1/p) = f(p,1/p) = f(0,1/p) = 0.$$

B.12. **Lemma** *There are canonical isomorphisms $\alpha_{M,N,L}\colon (M \otimes_R N) \otimes_R L \cong M \otimes_R (N \otimes_R L)$ and canonical isomorphisms $\tau_{M,N}\colon M \otimes_R N \to N \otimes_R M$. Furthermore, there are canonical isomorphisms $R \otimes_R M \cong M \cong M \otimes_R R$. These isomorphisms make $(\mathrm{Mod}(R), \otimes_R, R)$ into a* symmetric monoidal category*, that is, they satisfy various coherence axioms.*

Proof. The isomorphisms α and τ are inherited from the corresponding isomorphisms for the cartesian product. Finally, the scalar multiplication map $R \times M \to M$ is R-bilinear and satisfies the universal property of a tensor product. The coherence axioms for the τ is that $\tau_{M,N} \circ \tau_{N,M} = \mathrm{id}_{N \otimes_R M}$ for all N, M and that $\tau_{R,M}$ interchanges the two isomorphisms $R \otimes_R M \cong R$ and $M \otimes_R R \cong R$. There is also a coherence axiom for the interplay of τ and α. Furthermore, there is a coherence axiom for α involving 4 objects. Have a look at the wikipedia page for (symmetric) monoidal categories. All of these coherence axioms follow from the versions for the cartesian products. □

Recall that an adjunction of categories consists of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ together with a natural isomorphism

$$\tau\colon \mathrm{Hom}_{\mathcal{D}}(F(-), -) \cong \mathrm{Hom}_{\mathcal{C}}(-, G(-))\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Set}.$$

Given a functor $G\colon \mathcal{D} \to \mathcal{C}$, recall also that it *admits* a left adjoint if and only if for each $c \in \mathcal{C}$, the functor

$$\mathrm{Hom}_{\mathcal{C}}(c, G(-))\colon \mathcal{D} \to \mathrm{Set}$$

is corepresentable, i.e. isomorphic to $\mathrm{Hom}_{\mathcal{D}}(F(c), -)$ for some object $F(c) \in \mathcal{D}$ which is called a corepresenting object. If this is the case, choices of corepresenting objects $F(c)$ assemble into a functor $F\colon \mathcal{C} \to \mathcal{D}$ which is then left adjoint to G. This says that checking whether or not a given functor admits an adjoint is a "pointwise" question. Moreover, left adjoints commute with colimits and dually, right adjoints commute with limits. See chapter 6 in my lecture notes "Algebra" for further details. We will freely use these notions in what follows.

B.13. **Corollary** *Let R be a commutative ring and M an R-module. Then the functor* $\mathrm{Hom}_R(M, -)\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R)$ *admits a left adjoint* $M \otimes_R -\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R)$.

Proof. The bilinear map $M \times N \to M \otimes_R N$ part of the tensor product corresponds to a unique linear map $N \to \mathrm{Hom}_R(M, M \otimes_R N)$. Consider the composite

$$\mathrm{Hom}(M \otimes_R N, L) \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, M \otimes_R N), \mathrm{Hom}_R(M, L)) \longrightarrow \mathrm{Hom}_R(N, \mathrm{Hom}_R(M, L)).$$

Postcomposing the final term with the canonical bijection to $\mathrm{Hom}_{R,R}(M \times N, L)$ from Remark B.5, the composite becomes restriction along the bilinear map $M \times N \to M \otimes_R N$ and is therefore a bijection by the universal property of the tensor product. Consequently, the above composite is also a bijection, and natural in L by inspection. This precisely says that sending N to $M \otimes_R N$ assembles into a left adjoint of $\mathrm{Hom}_R(M, -)$. □

B.14. **Remark** Given a map $f\colon N \to N'$, the resulting map $\mathrm{id} \times f\colon M \times N \to M \times N' \to M \otimes_R N$ is R-bilinear and therefore extends uniquely to an R-linear map $\mathrm{id} \otimes f\colon M \otimes_R N \to M \otimes_R N'$. Unravelling the definitions, this map is indeed the effect of the functor $M \otimes_R -$ on the morphism f.

B.15. **Corollary** *Let R be a commutative ring and M an R-module. Then the functor* $M \otimes_R -\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R)$ *preserves colimits, and the functor* $\mathrm{Hom}_R(M, -)\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R)$ *preserves limits.*

Exercise. The functor $\mathrm{Hom}_R(-, M)\colon \mathrm{Mod}(R)^{\mathrm{op}} \to \mathrm{Mod}(R)$ also preserves limits.

B.16. **Remark** One says that a symmetric monoidal category is *closed* if for all objects M, the tensor product functor $M \otimes -$ admits a right adjoint. Consequently, $(\mathrm{Mod}(R), \otimes_R, R)$ is a closed symmetric monoidal category.

B.17. **Lemma** *Let $f\colon R \to S$ be a morphism in* CAlg[25] *and M an R-module. Then the R-modules $S \otimes_R M$ and $\mathrm{Hom}_R(S, M)$ are canonically the restriction of S-modules with the same name.*

Proof. We need to construct S-module structures on $S \otimes_R M$ and $\mathrm{Hom}_R(S, M)$ giving rise to the canonical R-module structures via the map f. We first, consider $S \otimes_R M$. For this we consider the following composite:

$$S \times (S \otimes_R M) \longrightarrow S \otimes_R (S \otimes_R M) \cong (S \otimes_R S) \otimes_R M \longrightarrow S \otimes_R M$$

where the last map is given by the multiplication map of S (note that it is R-bilinear). On elementary tensors, this map sends $(s, s' \otimes m)$ to $ss' \otimes m$. One then checks that this indeed defines an S-module structure on $S \otimes_R M$ whose restricted R-module structure is the canonical one since $r \cdot (s \otimes m) = rs \otimes m$ by definition of the tensor product. Likewise, we define a map $S \times \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S, M)$ by sending (s, f) to the map sf defined by $(sf)(s') = f(ss')$. Again, one checks that this is well-defined and gives an S-module structure on $\mathrm{Hom}_R(S, M)$. The restricted R-module structure is then the canonical one, since $(rf)(s) = f(rs) = r \cdot f(s)$ by R-linearity of f. □

B.18. **Proposition** *Let $R \to S$ be a map of commutative rings. Then the restriction of scalars functor* $\mathrm{Mod}(S) \to \mathrm{Mod}(R)$ *admits left and right adjoint, given by $S \otimes_R -$ and $\mathrm{Hom}_R(S, -)$.*

Proof. It remains to verify natural (in S-modules N and R-modules M) bijections

$$\mathrm{Hom}_S(S \otimes_R M, N) \cong \mathrm{Hom}_R(M, N) \quad \text{and} \quad \mathrm{Hom}_R(N, M) \cong \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)).$$

The first bijection is induced by sending a map $f\colon S \otimes_R M \to N$ to its restriction along $M \overset{\iota(1,-)}{\to} S \otimes_R M$. An inverse is given as follows: Let $g\colon M \to N$ be R-linear. Then the map $S \times M \to N$, $(s, m) \mapsto s \cdot g(m)$ is R-bilinear, and therefore descends to a map $S \otimes_R M \to N$, which, on elementary tensors sends $s \otimes m$ to $s \cdot g(m)$. This map is evidently S-linear. The second bijection for instance is given by sending $f\colon N \to M$ to the map $N \to \mathrm{Hom}_R(S, M)$, $n \mapsto (s \mapsto f(sn))$. Its inverse is given by postcomposing with the evaluation at 1 map $\mathrm{Hom}_R(S, M) \to M$ (sending g to $g(1)$). It is a direct check to see that these maps are natural in N and M. □

Exercise. Let M be an R-module and $\mathfrak{a}$ an ideal of R. There is a canonical isomorphism $R/\mathfrak{a} \otimes_R M \to M/\mathfrak{a}M$ of $R/\mathfrak{a}$-modules.

The terms appearing in the statement of the following lemma will be explained in the proof.

B.19. **Lemma** *Let $f\colon R \to S$ be a map of commutative rings. Then the extension of scalars functor* $\mathrm{Mod}(R) \to \mathrm{Mod}(S)$ *canonically admits the structure of a symmetric monoidal functor. In particular, given a commutative R-algebra A, the tensor product $S \otimes_R A$ is a commutative S-algebra.*

Proof. Giving the functor $S \otimes_R -$ a symmetric monoidal structure amounts to specifying isomorphisms $S \otimes_R R \cong S$ and $\rho_{M,N}\colon (S \otimes_R M) \otimes_S (S \otimes_R N) \to S \otimes_R (M \otimes_R N)$ compatible with the associativity and symmetry isomorphisms in $\mathrm{Mod}(R)$ and $\mathrm{Mod}(S)$, respectively, see again the Wikipedia page for the exact compatibilities that are required. For the first, we use the multiplication map $S \times R \to S$, $(s, r) \mapsto sf(r)$, note that it is R-bilinear and satisfies the

[25] The category of commutative rings and ring homomorphisms.

universal property of the tensor product. The isomorphism $\rho_{M,N}$ is given by the composite The isomorphism is given as follows:

$$\begin{aligned}(S \otimes_R M) \otimes_S (S \otimes_R N) &\cong (M \otimes_R S) \otimes_S (S \otimes_R M) \\ &\cong M \otimes_R (S \otimes_S S) \otimes_R M \\ &\cong M \otimes_R S \otimes_R M \\ &\cong S \otimes_R (M \otimes_R N).\end{aligned}$$

where all isomorphisms are associativity isomorphisms and symmetry isomorphisms (and the unitality isomorphism $S \otimes_S S \cong S$ we have also seen earlier). It then follows that $(S \otimes_R A)$ is a commutative ring with multiplication given by

$$(S \otimes_R A) \otimes_S (S \otimes_R A) \cong S \otimes_R (A \otimes_R A) \to S \otimes_R A$$

where the first is the isomorphism just discussed and the second is the multiplication map of A: Note that the multiplication map $A \times A \to A$ is R-bilinear since A is an R-algebra. Moreover, the R-algebra structure map $R \to A$ induces a ring map $S \cong S \otimes_R R \to S \otimes_R A$, making the latter an S-algebra. □

Exercise. The category of commutative R-algebras CAlg_R admits pushouts. A pushout of $B \leftarrow A \to C$ is given by $B \otimes_A C$.

B.20. **Definition** Let R be a commutative ring and M an R-module. Then M is called

(1) *flat*, if..
(2) *projective*, if..
(3) *free*, if...

Moreover, free modules are projective and projective modules are flat. See Exercise 4 Sheet 12.

B.21. **Definition** Let R be a commutative ring and M an R-module. A resolution of M is a non-negatively graded chain complex $R_\bullet$ together with a quasi-isomorphism $R_\bullet \to M$. Equivalently, if the chain complex

$$\cdots \to R_n \to \cdots \to R_1 \to R_0 \to M$$

is exact. Such a resolution is called flat, projective, or free if all modules R_n are flat, projective, or free, respectively. It is called of finite length if there exists $N \geq 0$ such that $R_n = 0$ for all $n \geq N$ and finite if it is of finite length and all terms R_n are finitely generated R-modules.

B.2. **Abelian categories.** We will now single out several properties the categories $\mathrm{Mod}(R)$ enjoy:

B.22. **Definition** An category is called pointed if it has an initial object $\emptyset$ and a terminal object $*$ and the canonical map $\emptyset \to *$ is an isomorphism. It is called semiadditive if it is pointed, has finite coproducts and products, and for all finite sets I and objects $\{X_i\}_{i\in I}$ the canonical map $\coprod_{i\in I} X_i \to \prod_{i\in I} X_i$[26] is an isomorphism. We then write $\bigoplus_{i\in I}$ for the formation of finite (co)products. It is called additive if it is semiadditive and all objects are grouplike, that is the canonical map $(\mathrm{pr}_1, \nabla)\colon X \oplus X \to X \oplus X$ is an isomorphism.

[26]Exercise: Think about what this canonical map is, and that its existence uses the fact that the category is pointed.

B.23. **Definition** Let $f\colon M \to N$ be a morphism in an additive category. Then $\ker(f) := \mathrm{Eq}(f,0)$, $\mathrm{coker}(f) := \mathrm{Coeq}(f,0)$, $\mathrm{Im}(f) := \ker(N \to \mathrm{coker}(f))$, $\mathrm{Coim}(f) := \mathrm{coker}(\ker(f) \to M)$.

B.24. **Definition** A category is called abelian if it is additive and satisfyies the following further properties:

(1) every morphism has a kernel and cokernel, and
(2) the evident map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism.

B.25. **Remark** Standard definitions say that (2) above is replaced by the following condition. Let $f\colon M \to N$ be a morphism. If f is a monomorphism, the canonical map $M \to \mathrm{Im}(f)$ is an isomorphism, and if f is an epimorphism, the canonical map $\mathrm{Coim}(f) \to N$ is an isomorphims.

That our definition implies this condition follows easily from the observation that $\mathrm{coker}(f) = 0$ if f is an epimorphism and $\ker(f) = 0$ if f is a monomorphism. The converse is left as an exercise, but see e.g. [Pro, §12.5].

B.26. **Example** Let R be a ring. Then $\mathrm{Mod}(R)$ is an abelian category. Indeed, $\mathrm{Mod}(R)$ is additive and bicomplete, and property (2) in Definition B.24 is a consequence of the homomorphism theorems (perhaps it is called the 2nd isomorphism theorem?!).

B.27. **Example** Let $\mathcal{A}$ be an abelian category and K a small category. Then the category $\mathrm{Fun}(K, \mathcal{A})$ is an abelian category.

B.28. **Definition** An object X of an abelian category is called projective, if $\mathrm{Hom}_{\mathcal{A}}(X, -)$ is an exact functor, or equivalently, if for all maps $f\colon K \to L$ with $\mathrm{coker}(f) = 0$ and maps $X \to L$, there exists a lift $X \to K$ along f.

B.29. **Definition** Let $\mathcal{A}$ be an abelian category. Then $\mathrm{Ch}(\mathcal{A})$ is the category of chain complexes in $\mathcal{A}$. Homology is defined as we are used to: $H_n(C_\bullet) = \ker(d_n)/\mathrm{Im}(d_{n+1})$. This is indeed an object of $\mathcal{A}$ by the axioms of abelian categories.

B.30. **Remark** For $\mathcal{A}$ an abelian category, we have that $\mathrm{Ch}(\mathcal{A})$ is again abelian. In particular, one may formally form $\mathrm{Ch}(\mathrm{Ch}(\mathcal{A}))$. Objects therein are also called double complexes.

B.31. **Definition** The notion of quasi-isomorphisms, chain homotopies extend from $\mathrm{Ch}(R)$ to $\mathrm{Ch}(\mathcal{A})$ verbatim.[27] The notion of projective resolutions/projective chain complexes extends verbatim.

The following are two important lemmata whose proofs go under the name of *diagram chasing*. Both apply to a general abelian category $\mathcal{A}$, but it turns out that it is sufficient to prove them in the case $\mathcal{A} = \mathrm{Mod}(R)$.

[27] Note that in any abelian category, $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is an abelian group, so there is a well-defined notion of sum and difference of maps $f, g\colon X \to Y$.

B.32. **Lemma** (Snake Lemma) *Let $\mathcal{A}$ be an abelian category and*

$$\begin{array}{ccccccc} (0 \longrightarrow\ )M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 \longrightarrow N' & \longrightarrow & N & \longrightarrow & N''(& \longrightarrow & 0) \end{array}$$

a commutative diagram. Then there exists a natural map $\partial \ker(f'') \to \mathrm{coker}(f')$ *making the sequence*

$$(0 \to) \ker(f') \to \ker(f) \to \ker(f'') \xrightarrow{\partial} \mathrm{coker}(f') \to \mathrm{coker}(f) \to \mathrm{coker}(f'')(\to 0)$$

exact.

B.33. **Lemma** (5 Lemma) *Let $\mathcal{A}$ be an abelian category and let*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

be a commutative diagram.

(1) *If b and e are surjective and e is injective, then c is surjective.*
(2) *if b and e are injective and a is surjective, then c is injective.*
(3) *If b and e are isomorphisms, a is surjective and e is injective, then c is an isomorphism.*

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